Special Functions

&

Complex Variable

Series solution to the differential equations

Motivation for series solution:-

The factors that motivate the use of series solutions are
1) Series solutions are of great importance in determining the solutions for second order differential equations.
2) These solutions facilitate series expansions and generate several new functions of different class.
3) It is a standard method for solving initial value problems with variable coefficients (especially in their closed form).

Power series:-

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0(x-x_0)^0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

$$\therefore \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots \quad \text{--- (1)}$$

where $a_0, a_1, a_2, \ldots$ are real constants is called power series in powers of $(x-x_0)$.

If $x_0 = 0$ the power series become

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \quad \text{--- (2)}$$

If $x = x_0$ in eq^n (1), the power series (1) is always convergent.

If $x = 0$ in eq^n (2), the power series (2) is always convergent.

If $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = R$ exist the power series ① is convergent $\forall x$, such that $|x-x_0|<R$.

$$|x-x_0| < R$$

$$-R < x-x_0 < R$$

$$-R+x_0 < x < R+x_0$$

$$\therefore x \in (x_0-R, x_0+R)$$

In this case R is said to be radius of converges of the power series.

If $R=\infty$, the power series converges for all values of x and we can say that the power series has infinite radius of converges.

The interval (x_0-R, x_0+R) is said to be interval of converges for the series eqⁿ ①.

The interval $(-R, R)$ is said to be the interval of conver-ges for the series ②

Analytic function:-

Let a function $f(x)$ be derivable at every point x in ε neighbourhood of x_0 i.e, $f'(x)$ exist $\forall x$ such that $|x-x_0|<\varepsilon$ where $\varepsilon>0$, then $f(x)$ is said to be analy-tic at x_0.

Series solutions to differential equations:-

consider the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \quad \text{——} \; (1) \text{ say}$$

where $P(x)$ and $R(x)$ are functions of x alone. The validity and the conditions for the differential eq^n (1) to have the series solution can be determined.

Ordinary point :-

consider $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$

If the coefficient of $\frac{d^2y}{dx^2} \neq 0$ at $x=a$, i.e., $P(x) \neq 0$ at $x=a$ then $x=a$ is called an ordinary point of differential eq^n (1)

Singular point :-

If $P(x)=0$ at $x=a$, then $x=a$ is called singular point.

Ex:-

consider the differential equation

$(1-x^2)y'' - 2xy' + P(P+1)y = 0$ —— (1) [It is legendre's eq^n]

compare eq^n (1) with $P(x)y'' + Q(x)y' + R(x)y = 0$

Here $P(x) = 1-x^2$, $Q(x) = -2x$ $\quad R(x) = P(P+1)$

Let $P(x)=0$ then $1-x^2 = 0$

$$x^2 = 1$$

$$x = \pm 1$$

$x = \pm 1$ are called singular points.

and $x = 0, 2, 3, \ldots$ are called ordinary points of given differential equation.

Types of Singular points:-

The singular points are two types

1. Regular singular point

2. Irregular singular point.

Regular singular point :-

consider the differential eq^n $p(x)y'' + Q(x)y' + R(x)y = 0$ —— ①

Eq^n ① can be written as $y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$

$$y'' + P_1(x)y' + P_2(x)y = 0 \text{ —— } ②$$

where $P_1(x) = \frac{Q(x)}{P(x)}$, $P_2(x) = \frac{R(x)}{P(x)}$ $(P(x) \neq 0)$

A Singular point $x=a$ of eq^n ② is called regular sin-gular point if both $(x-a)P_1(x)$ and $(x-a)^2 P_2(x)$ are analytic at the point $x=a$

A singular point is said to be irregular if it is not regular and the differential eq^n ① has no series solu-tion.

Ex:-

consider the differential eq^n $x^3(2-x)^2 \frac{d^2y}{dx^2} - 2x^2(2-x)\frac{dy}{dx} + 3y = 0$ —— ①

comparing this with $p(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$

$$P(x) = x^3(2-x)^2 \quad Q(x) = -2x^2(2-x) \quad R(x) = 3$$

$P(x)=0 \Rightarrow x^3(2-x)^2=0$

$x^3=0, \ (2-x)^2=0$

$x=0, 2$

$x=0$ and 2 are singular while all other points are ordinary points.

Divide given differential eq^n by $x^3(2-x)^2$.

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{2x^2(2-x)}{x^3(2-x)^2}\frac{dy}{dx} + \frac{3y}{x^3(2-x)^2} = 0$$

$$\frac{d^2y}{dx^2} - \frac{2}{x(2-x)}\frac{dy}{dx} + \frac{3}{x^3(2-x)^2}y = 0 \quad \text{—— (2)}$$

compare the above differential eq^n with $y'' + P_1(x)y' + P_2(x)y = 0$

$$P_1(x) = \frac{-2}{x(2-x)} \qquad P_2(x) = \frac{3}{x^3(2-x)^2}$$

case (i): At the point $x=0$

$$(x-a)P_1(x) = x\left(\frac{-2}{x(2-x)}\right) = \frac{2}{x-2}$$

$$(x-a)^2 P_2(x) = x^2\left(\frac{3}{x^3(2-x)^2}\right) = \frac{3}{x(2-x)^2}$$

$xP_1(x)$ is analytic at $x=0$

$x^2P_2(x)$ is not analytic at $x=0$

$\therefore x=0$ is irregular singular point (not regular).

case 2: At the point $x=2$.

$$(x-a)P_1(x) = (x-2)\left(\frac{-2}{x(2-x)}\right) = \frac{2}{x}$$

$$(x-a)^2 P_2(x) = (x-2)^2\left(\frac{3}{x^3(2-x)^2}\right) = \frac{3}{x^3}$$

$$\therefore (x-2)P_1(x) = \frac{2}{x} \quad \& \quad (x-2)^2 P_2(x) = \frac{3}{x^3}$$

$(x-2)P_1(x)$ is analytic at $x=2$ and $(x-2)^2 P_2(x)$ is analytic at $x=2$.

$\therefore x=2$ is Regular singular point.

Find the Regular and singular points of the differential eq^{n}s

i) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

ii) $x^2y'' + axy' + by = 0$, a,b are constant.

i) sol^n: Given $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ —— ①

comparing this with $P(x)y'' + Q(x)y' + R(x)y = 0$

$P(x) = 1-x^2 \qquad Q(x) = -2x \qquad R(x) = n(n+1)$

$$P(x) = 0 \Rightarrow 1-x^2 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

$x = -1$ and 1 are singular while all other points are ordinary points.

Divide given differential eq^n by $(1-x^2)$

$$\Rightarrow y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{(1-x^2)}y = 0 \quad \text{—— ②}$$

compare the above differential eq^n with $y''+P_1(x)y'+P_2(x)y=0$

$$P_1(x)=\frac{-2x}{1-x^2}, \quad P_2(x)=\frac{n(n+1)}{1-x^2}$$

case(i): At the point $x=-1$

$$(x-a)P_1(x)=(x+1)\left(\frac{-2x}{1-x^2}\right)$$

$$(x-a)^2P_2(x)=(x+1)^2\left(\frac{n(n+1)}{1-x^2}\right)$$

$(x+1)\left(\frac{-2x}{1-x^2}\right)$ is not analytic at $x=-1$

$(x+1)^2\left(\frac{n(n+1)}{1-x^2}\right)$ is not analytic at $x=-1$.

$\therefore$ $x=-1$ is irregular singular point (not regular)

case (ii): At the point $x=1$

$$(x-a)P_1(x)=(x-1)\left(\frac{-2x}{1-x^2}\right)$$

$$(x-a)^2P_2(x)=(x-1)^2\left(\frac{n(n+1)}{1-x^2}\right)$$

$(x-1)\left(\frac{-2x}{1-x^2}\right)$ is not analytic at $x=1$

$(x-1)^2\left(\frac{n(n+1)}{1-x^2}\right)$ is not analytic at $x=1$

$\therefore$ $x=1$ is irregular singular point (not regular).

ii) sol^n: Given $x^2y''+axy'+by=0$ — ①

comparing this eq^n with $P(x)y''+Q(x)y'+R(x)y=0$

$P(x)=x^2 \qquad Q(x)=ax \qquad R(x)=b$

$$P(x) = 0 \Rightarrow x^2 = 0$$
$$x = 0$$

$x=0$ is singular while all other points are ordinary points.

Divide given differential eqn by x^2

$$y'' + \frac{ax}{x^2} y' + \frac{b}{x^2} y = 0 \quad \text{--- (2)}$$

compare the above differential eqn with $y'' + P_1(x) y' + P_2(x) y = 0$

$$P_1(x) = \frac{ax}{x^2} = \frac{a}{x} \qquad P_2(x) = \frac{b}{x^2}$$

At point $x=0$:

$$(x-a) P_1(x) = x\left(\frac{a}{x}\right) = a$$

$$(x-a)^2 P_2(x) = x^2 \left(\frac{b}{x^2}\right) = b$$

$x P_1(x)$ is analytic at $x=0$

$x^2 P_2(x)$ is analytic at $x=0$

$\therefore$ $x=0$ is regular singular point.

Method - 1:

The power series method to solve an ordinary differential equation.

The power series method is a very efficient standard procedure used to solve linear differential equations with variable coefficients, this method helps solutions in the form of power series.

Power series solution about the ordinary point $x=0$:-

Working rule to solve the differential equation

$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$:-

Step-1: Given differential equation is $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$ —— (1)

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials in 'x' and $P(x) \neq 0$ at $x=0$

Assume that the solution of eqn (1) to be of the form

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$ —— (2)

where $a_0, a_1, a_2, \dots$ are constants to be found.

Step-2: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by using eqn (2) and substitute the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in eqn (1). The result of this substitution is an identity.

Step-3: Equate to zero the coefficients of the various powers of x. Now we will get a number of equations involving $a_0, a_1, a_2, \dots$

The result obtained by equating the coefficient of x^n to zero is called recurrence relation and it can be used to compute additional constants.

Step-4: Determine the values of $a_2, a_3, a_4, \dots$ in terms of a_0 and a_1

Step-5: Finally substitute the values of $a_2, a_3, a_4, \dots$ in eqn (2) to get the desired series solution involving two arbitrary constants a_0 and a_1.

Problems:

1) Solve in series the equation $\frac{d^2y}{dx^2} - xy = 0$ about $x=0$.

Sol^n: Given differential equation is $\frac{d^2y}{dx^2} - xy = 0$ —— ① say

Let the series solution of eq^n ① be of the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \quad \text{—— ② say}$$

Now we have to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + n \cdot a_n x^{n-1} + \cdots \quad \text{—— ③}$$

and $$\frac{d^2y}{dx^2} = 2a_2 + 2\cdot3\cdot a_3x + 3\cdot4\cdot a_4\cdot x^2 + 4\cdot5 a_5 x^3 + \cdots + (n-1)n a_n \cdot x^{n-2} + \cdots \quad \text{—— ④}$$

Sub eq^n ② and ④ in eq^n ①

$$① \Rightarrow (2a_2 + 3\cdot2 a_3x + 4\cdot3 a_4x^2 + 5\cdot4 a_5x^3 + \cdots + n(n-1)a_nx^{n-2}) -$$
$$x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots) = 0$$

$$\Rightarrow 2a_2 + (3\cdot2a_3 - a_0)x + (4\cdot3a_4 - a_1)x^2 + (5\cdot4a_5 - a_2)x^3 + \cdots +$$
$$((n+2)(n+1)a_{n+2} - a_{n-1})x^n + \cdots = 0 \quad \text{—— ⑤}$$

Now equating to zero the coefficients of various powers x.

$$2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$$

$$3\cdot2a_3 - a_0 = 0 \Rightarrow 6a_3 = a_0 \Rightarrow \boxed{a_3 = \frac{a_0}{6} = \frac{a_0}{3!}}$$

$$4\cdot3a_4 - a_1 = 0 \Rightarrow \boxed{a_4 = \frac{a_1}{12} = \frac{2a_1}{4!}}$$

$5\cdot 4 a_5 - a_2 = 0$

$a_5 - \frac{a_2}{20} = 0 \Rightarrow \boxed{a_5 = 0}$ and so on.

$(n+2)(n+1)a_{n+2} - a_{n-1} = 0$

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

The recurrence relation is $\boxed{a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}}$ — (6)

Putting $n = 4, 5, 6, \ldots$ we get

$$a_6 = \frac{a_3}{6\cdot 5} = \frac{1}{6\cdot 5}\cdot\frac{a_0}{3!} = \frac{4a_0}{6!}$$

$$a_7 = \frac{a_4}{7\cdot 6} = \frac{1}{7\cdot 6}\cdot\frac{2a_1}{4!} = \frac{10}{7!}a_1$$

$$a_8 = \frac{a_5}{8\cdot 7} = 0 \quad [\because a_5 = 0]$$

$\vdots$

Sub all the values in eq^n (2)

(2) $\Rightarrow y = a_0 + a_1 x + 0\cdot x^2 + \frac{a_0}{3!}\cdot x^3 + \frac{2a_1}{4!}x^4 + 0\cdot x^5 + \frac{4a_0}{6!}x^6 + \frac{10}{7!}a_1 x^7 + 0\cdot x^8 + \cdots$

$$y = a_0 + a_1 x + \frac{a_0}{3!}x^3 + \frac{2a_1}{4!}x^4 + \frac{4a_0}{6!}x^6 + \frac{10}{7!}a_1 x^7 + \cdots$$

$$y = \left(1 + \frac{1}{3!}x^3 + \frac{4}{6!}x^6 + \cdots\right)a_0 + \left(x + \frac{2}{4!}x^4 + \frac{10}{7!}x^7 + \cdots\right)a_1$$

which is the general solution of given second order differential equation as it contains two arbitrary constants

1) Find the power series solution of the equation $y''+xy'+y=0$ in powers of x.

(or)

Solve in series the equation $y''+xy'+y=0$ about $x=0$.

Soln: Given differential equation is $y''+xy'+y=0$ — ① say

Let the series solution of eqn ① be of the form

$$y=a_0+a_1x+a_2x^2+a_3x^3+\ldots \quad \text{— ② say}$$

Now we have to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$\frac{dy}{dx}=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots+na_nx^{n-1}+\ldots \quad \text{— ③}$$

$$\frac{d^2y}{dx^2}=2a_2+2\cdot3a_3x+3\cdot4a_4x^2+4\cdot5a_5x^3+\ldots+(n-1)na_nx^{n-2}+\ldots \quad \text{— ④}$$

sub eqn ②, ③ and ④ in eqn ①

$$① \Rightarrow (2a_2+2\cdot3a_3x+3\cdot4a_4x^2+4\cdot5a_5x^3+\ldots+(n-1)n\cdot a_nx^{n-2})+$$
$$x[a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots+na_nx^{n-1}+\ldots]+a_0+a_1x+a_2x^2$$
$$+a_3x^3+\ldots=0$$

$$\Rightarrow [2a_2+a_0+x(3\cdot2a_3+a_1]$$

$$(2a_2+3\cdot2a_3x+4\cdot3a_4x^2+5\cdot4a_5x^3+\ldots+(n-1)n\cdot a_nx^{n-2}+\ldots+(n+1)(n+2)\cdot$$
$$a_{n+2}\cdot x^n)+x[a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots+na_nx^{n-1}+(n+1)a_{n+1}x^n]+$$
$$+a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n+\ldots=0$$

$$\Rightarrow [2a_2+a_0]+[3\cdot2a_3+a_1+a_1]x+[4\cdot3a_4+2a_2+a_2]x^2+[5\cdot4a_5+$$
$$3a_3+a_3]x^3+\ldots+[(n+1)(n+2)a_{n+2}+na_n+a_n]x^n=0 \quad \text{— ⑤}$$

Now equating to zero the coefficients of various powers x.

$2a_2 + a_0 = 0 \rightarrow 2a_2 = -a_0 \Rightarrow \boxed{a_2 = -\frac{a_0}{2}}$

$3 \cdot 2a_3 + a_1 + a_1 = 0 \Rightarrow 6a_3 + 2a_1 = 0 \Rightarrow 6a_3 = 2a_1 \Rightarrow \boxed{a_3 = \frac{-a_1}{3}}$

$4 \cdot 3a_4 + 2a_2 + a_2 = 0 \Rightarrow 12a_4 + 3a_2 = 0 \Rightarrow 12a_4 = -3a_2 \Rightarrow a_4 = -\frac{a_2}{4}$

$a_4 = -\frac{1}{4}\left(-\frac{a_0}{2}\right) = \frac{a_0}{8}$

$\boxed{a_4 = \frac{a_0}{8}}$

$5 \cdot 4a_5 + 3a_3 + a_3 = 0 \Rightarrow 20a_5 + 4a_3 = 0 \Rightarrow 20a_5 = -4a_3$

$a_5 = -\frac{4}{20} a_3 = -\frac{1}{5}\left(\frac{-a_1}{3}\right)$

$\boxed{a_5 = \frac{+a_1}{15}}$ and so on

$(n+1)(n+2)a_{n+2} + na_n + a_n = 0$

$(n+1)(n+2)a_{n+2} + (n+1)a_n = 0$

$\cancel{(n+1)}(n+2)a_{n+2} = \frac{-\cancel{(n+1)}a_n}{}$

$a_{n+2} = \frac{-a_n}{n+2}$

Sub all the values in eq^n (2)

(2) $\Rightarrow y = a_0 + a_1 x + \left(-\frac{a_0}{2}\right)x^2 + \left(\frac{a_1}{3}\right)x^3 + \left(\frac{a_0}{8}\right)x^4 + \left(-\frac{a_1}{15}\right)x^5 + \dots$

$y = a_0 + a_1 x - \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{8}x^4 - \frac{a_1}{15}x^5 + \dots$

$y = \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \dots\right)a_0 + \left(x - \frac{x^3}{3} + \frac{x^5}{15} + \dots\right)a_1$

which is the general sol^n of given second order differential equation as it contains two arbitrary constants.

3) Find the solution of $\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 4xy = x^2+2x+4$ (about $x=0$).

Sol^n: Given differential equation is

$$\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} + 4xy = x^2+2x+4 \quad \text{—— (1)}$$

Assume that the solution of eq^n (1) be of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots \quad \text{—— (2)}$$

Now, $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots \quad \text{—— (3)}$$

$$\frac{d^2y}{dx^2} = 2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \dots + n\cdot(n-1)a_nx^{n-2} + \dots \quad \text{—— (4)}$$

Sub the eq^{n}s (2), (3) and (4) in eq^n (1)

$$(1) \Rightarrow (2a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + \dots + n\cdot(n-1)a_nx^{n-2} + \dots) - 2x^2(a_1 + 2a_2x$$

$(n+2)(n+1)a_{n+2}\cdot x^n$ $\quad -2x^2((n-1)a_{n-1}x^{n-2})$

$$+ 3a_3x^2 + \dots + na_nx^{n-1} + \dots) + 4x(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots)$$

$4x(a_{n-1}\cdot x^{n-1}) = 4a_{n-1}x^n$

$$= x^2+2x+4$$

$$\Rightarrow 2a_2 + (3\cdot 2a_3 + 4a_0)x + (12a_4 - 2a_1 + 4a_1)x^2 + \dots + ((n+2)(n+1)a_{n+2}$$

$$- 2(n-1)a_{n-1} + 4a_{n-1})x^n = x^2+2x+4$$

Equating the coefficients of x, x^2 and x^3 on both sides we get

$$2a_2 = 4 \Rightarrow \boxed{a_2 = 2}$$

$$6a_3 + 4a_0 = 2 \Rightarrow 6a_3 = 2 - 4a_0 \Rightarrow a_3 = \frac{2-4a_0}{6}$$

$$\boxed{a_3 = \frac{1-2a_0}{3}}$$

$12a_4 + 2a_1 = 1$

$12a_4 = 1 - 2a_1$

$$\boxed{a_4 = \frac{1-2a_1}{12}}$$

Now equate the coefficient of x^n on b.s

$(n+2)(n+1)a_{n+2} + 2(n-1)a_{n-1} + 4a_{n-1} = 0$

$(n+2)(n+1)a_{n+2} + (2n+6)a_{n-1} = 0$

$(n+2)(n+1)a_{n+2} = -2(n+1)a_{n-1}$

$a_{n+2} = -\frac{2a_{n-1}}{n+2}$

$(n+2)(n+1)a_{n+2} = (2n-6)a_{n-1}$

$$\boxed{a_{n+2} = \frac{2(n-3)}{(n+2)(n+1)}a_{n-1}} \quad —(5)$$

which is the recurrence relation

putting $n = 3, 4, 5, \ldots$ in eq^n (5)

$n=3, \quad a_5 = 0$

$n=4, \quad a_6 = \frac{2}{30}a_3 = \frac{1}{15}\left(\frac{1-2a_0}{3}\right)$

$a_6 = \frac{1-2a_0}{45}$

putting the values in eq^n (2) we get

$y = a_0 + a_1x + 2x^2 + \left(\frac{1-2a_0}{3}\right)x^3 + \left(\frac{1-2a_1}{12}\right)x^4 + \left(\frac{1-2a_0}{45}\right)x^6 + \ldots$

$y = a_0 + a_1x + 2x^2 + \left(\frac{1}{3} - \frac{2a_0}{3}\right)x^3 + \left(\frac{1}{12} - \frac{a_1}{6}\right)x^4 + \ldots$

$y = \left(1 - \frac{2}{3}x^3\right)a_0 + (x -$

$y = \left(1 - \frac{2}{3}x^3 + \ldots\right)a_0 + \left(x - \frac{x^4}{6} + \ldots\right)a_1 + \left(2x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \ldots\right)$

which is the required solⁿ of given differential eqⁿ.

4) solve in series $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$.

Solⁿ: Given differential eqⁿ is $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$ —— ①

clearly $x=0$ is ordinary point

Assume that the solⁿ of eqⁿ ① be of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots \quad \text{—— ②}$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots \quad \text{— ③}$$

$$\frac{d^2y}{dx^2} = 2a_2 + 3\cdot 2 a_3 x + 12 a_4 x^2 + \dots + n(n-1)a_n x^{n-2} + \dots \quad \text{—— ④}$$

Sub the eqⁿ ②, ③ and ④ in eqⁿ ①

$$① \Rightarrow (1-x^2)(2a_2 + 6a_3 x + 12a_4 x^2 + \dots + n(n-1)a_n x^{n-2} + \dots) - x(a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} + \dots) + 4(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots) = 0$$

$$\Rightarrow (2a_2 + 6a_3 x + 12a_4 x^2 + \dots + n(n-1)a_n x^{n-2} + \dots) + (-2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 - \dots - n(n-1)a_n x^n + \dots) + (-a_1 x - 2a_2 x^2 - 3a_3 x^3 - \dots - n a_n x^n - \dots) + (4a_0 + 4a_1 x + 4a_2 x^2 + 4a_3 x^3 + \dots + 4a_n x^n + \dots) = 0$$

$$\Rightarrow (2a_2 + 4a_0) + (6a_3 - a_1 + 4a_1)x + (12a_4 - \cancel{2a_2} - \cancel{2a_2} + \cancel{4a_2})x^2 + (20a_5 - 6a_3 - 3a_3 + 4a_3)x^3 + \dots + ((n+2)(n+1)a_{n+2} - n(n-1)a_n - n a_n + 4a_n)x^n = 0$$

$$\Rightarrow (2a_2 + 4a_0) + (6a_3 + 3a_1)x + 12a_4 x^2 + (20a_5 - 5a_3)x^3 + \dots + ((n+2)(n+1)a_{n+2} + (4-n^2)a_n)x^n = 0$$

Equating the coefficients of various powers of x to zero. We get

$2a_2 + 4a_0 = 0 \Rightarrow 2a_2 = -4a_0 \Rightarrow \boxed{a_2 = -2a_0}$

$6a_3 + 3a_1 = 0 \Rightarrow 6a_3 = -3a_1 \Rightarrow \boxed{a_3 = -\frac{a_1}{2}}$

$12a_4 = 0 \Rightarrow \boxed{a_4 = 0}$

$20a_5 - 5a_3 = 0 \Rightarrow 20a_5 = 5a_3 \quad \therefore a_5 = \frac{a_3}{4}$

$a_5 = \frac{1}{4}\left(-\frac{a_1}{2}\right) \Rightarrow \boxed{a_5 = -\frac{a_1}{8}}$

$(n+2)(n+1)a_{n+2} + (4-n^2)a_n = 0$

$(n+2)(n+1)a_{n+2} = (n^2-4)a_n$

$$a_{n+2} = \frac{\cancel{(n+2)}(n-2)}{\cancel{(n+2)}(n+1)} a_n$$

$$\boxed{a_{n+2} = \frac{n-2}{n+1} \cdot a_n} \quad \text{—— (5)}$$

$\therefore$ The recurrence relation is $a_{n+2} = \frac{n-2}{n+1} \cdot a_n$

putting $n = 4, 5, 6, \dots$ in eqⁿ (5), we get

$n=4, \quad a_6 = \frac{2}{5} \cdot a_4 = 0 \quad [\because a_4 = 0]$

$n=5, \quad a_7 = \frac{3}{6} \cdot a_5 = \frac{3}{6}\left(-\frac{a_1}{8}\right) = -\frac{1}{16}a_1$

$\boxed{a_7 = -\frac{1}{16}a_1}$

$n=6, \quad a_8 = \frac{4}{7} \cdot a_6 = 0 \quad [\because a_6 = 0]$

$n=7, \quad a_9 = \frac{5}{8} \cdot a_7 = \frac{5}{8}\left(-\frac{1}{16}\right)a_1 = -\frac{5}{128}a_1$

$\boxed{a_9 = -\frac{5}{128}a_1}$ and so on

Sub the values of $a_2, a_3, a_4, \ldots$ in eq^n (2)

$$(2) \Rightarrow y = a_0 + a_1x + (-2a_0)x^2 + \left(-\frac{1}{2}a_1\right)x^3 + (0)x^4 + \left(-\frac{1}{8}a_1\right)x^5 + 0x^6 + \left(-\frac{1}{16}a_1\right)x^7 + 0x^8 + \left(-\frac{5}{128}a_1\right)x^9 + \ldots$$

$$y = (1 - 2x^2 + \ldots)a_0 + \left(x - \frac{x^3}{2} - \frac{x^5}{8} - \frac{x^7}{16} - \frac{5x^9}{128} + \ldots\right)a_1$$

which is the required sol^n of given differential eq^n.

5) Solve in series the eq^n $\frac{dy}{dx} + 2xy = \frac{1}{1-x}$ about $x=0$.

Sol^n: given differential eq^n is $\frac{dy}{dx} + 2xy = \frac{1}{1-x}$ —(1)

clearly $x=0$ is ordinary point

Assume that the sol^n of eq^n (1) be of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \quad \text{—(2)}$$

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + na_nx^{n-1} + \ldots \quad \text{—(3)}$$

Sub the eq^{n}s (2) and (3) in eq^n (1)

$$(1) \Rightarrow (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + na_nx^{n-1} + \ldots) + 2x(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots) = \frac{1}{1-x}$$

$$\Rightarrow (a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + na_nx^{n-1} + (n+1)a_{n+1} \cdot x^n + \ldots) + (2a_0x + 2a_1x^2 + 2a_2x^3 + \ldots + 2a_{n-1}x^n + 2a_nx^{n+1} + \ldots) = \frac{1}{1-x}$$

$$\Rightarrow a_1 + (2a_2 + 2a_0)x + (3a_3 + 2a_1)x^2 + (4a_4 + 2a_2)x^3 + \ldots + ((n+1)a_{n+1} + 2a_{n-1})x^n = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

Equating on both sides.

$\Rightarrow \boxed{a_1 = 1}$

$2a_2 + 2a_0 = 1 \Rightarrow 2a_2 = 1 - 2a_0 \Rightarrow \boxed{a_2 = \frac{1-2a_0}{2}}$

$3a_3 + 2a_1 = 1 \Rightarrow 3a_3 = 1 - 2a_1 \Rightarrow 3a_3 = 1 - 2 = -1$

$\boxed{a_3 = -1/3}$

$4a_4 + 2a_2 = 1 \Rightarrow 4a_4 = 1 - 2a_2$

$a_4 = \frac{1-2a_2}{4} = \frac{1}{4}\left(1 - \cancel{2}\left(\frac{1-2a_0}{\cancel{2}}\right)\right)$

$a_4 = \frac{1}{4}\left(\cancel{1} - \cancel{1} + 2a_0\right) = \frac{2a_0}{4}$

$\boxed{a_4 = \frac{a_0}{2}}$

$\vdots$

$(n+1)a_{n+1} + 2a_{n-1} = 1$

$(n+1)a_{n+1} = 1 - 2a_{n-1}$

$\boxed{a_{n+1} = \frac{1 - 2a_{n-1}}{n+1}}$ —— (4)

which is the recurrence relation

putting $n = 4, 5, 6, \ldots$ in eq^n (4)

$n = 4, \quad a_5 = \frac{1-2a_3}{5} = \frac{1 - 2\left(-\frac{1}{3}\right)}{5} = \frac{1 + \frac{2}{3}}{5}$

$\boxed{a_5 = \frac{1}{3}}$

$n = 5, \quad a_6 = \frac{1 - 2a_4}{6} = \frac{1 - \cancel{2}\left(\frac{a_0}{\cancel{2}}\right)}{6} = \frac{1 - a_0}{6}$

$\boxed{a_6 = \frac{1 - a_0}{6}}$

Sub the values of $a_1, a_2, a_3, \ldots$ in eq^n (2)

$$(2) \Rightarrow y = a_0 + 1(x) + \left(\frac{1-2a_0}{2}\right)x^2 + \left(-\frac{1}{3}\right)x^3 + \left(\frac{a_0}{2}\right)x^4 + \left(\frac{1}{3}\right)x^5 + \left(\frac{1-a_0}{6}\right)x^6 + \ldots$$

$$y = \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \ldots\right)a_0 + \left(x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^5}{3} + \frac{x^6}{6} + \ldots\right)$$

which is the required sol^n for given differential eq^n.

Series solution when $x = 0$ is a regular singular point - Frobenius method.

If $x = 0$ is a regular point, we shall use the frobenius method for finding series solution about $x = 0$.

Working procedure for frobenius method:-

consider the differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$ — (1)

equate the coefficient of y'' to zero i.e, $P(x) = 0$

$P(x) = 0$ when $x = a$ then $x = a$ is called singular point of the given differential eq^n

Assume that $x = 0$ is a regular singular point of the given d.e (1)

Step 1: Divide eq^n (1) with $P(x)$ and the resultant eq^n is of the form $y'' + P_1(x)y' + P_2(x)y = 0$ — (2)

where $P_1(x) = \dfrac{Q(x)}{P(x)}$ $\quad P_2(x) = \dfrac{R(x)}{P(x)}$

$(x-a)P_1(x)$ and $(x-a)^2P_2(x)$ are analytic at $x = a$.

$\therefore x = a$ is called regular singular point (If $xP_1(x)$ &

Sub the values of $a_1, a_2, a_3 \ldots$ in eq^n (2)

$$(2) \Rightarrow y = a_0 + 1(x) + \left(\frac{1-2a_0}{2}\right)x^2 + \left(\frac{-1}{3}\right)x^3 + \left(\frac{a_0}{2}\right)x^4 + \left(\frac{1}{3}\right)x^5 + \left(\frac{1-a_0}{6}\right)x^6 + \ldots$$

$$y = \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \ldots\right)a_0 + \left(x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^5}{3} + \frac{x^6}{6} + \ldots\right)$$

which is the required sol^n for given differential eq^n.

Series solution when $x=0$ is a regular singular point - Frobenius method.

If $x=0$ is a regular point, we shall use the frobenius method for finding series solution about $x=0$.

Working procedure for frobenius method:-

consider the differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$ — (1)

Equate the coefficient of y'' to zero i.e, $P(x)=0$

$P(x)=0$ when $x=a$ then $x=a$ is called singular point of the given differential eq^n.

-Assume that $x=0$ is a regular singular point of the given d.e (1)

Step-1: Divide eq^n (1) with $P(x)$ and the resultant eq^n is of the form $y'' + P_1(x)y' + P_2(x)y = 0$ — (2)

where $P_1(x) = \frac{Q(x)}{P(x)}$ $\quad P_2(x) = \frac{R(x)}{P(x)}$

$(x-a)P_1(x)$ and $(x-a)^2P_2(x)$ are analytic at $x=a$.

$\therefore x=a$ is called regular singular point (If $xP_1(x)$ &

$x^2 P_2(x)$ are analytic at $x=0$ then $x=0$ is called regular singular point).

Step-2: Assume that the solution of the differential eq^n is

$$y = x^m(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots),\ a_0 \neq 0 \quad \text{--- (3) say}$$

Step-3: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Sub the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq^n (1)

Step-4: Equate the coefficient of the lowest degree in x to zero. We obtain a quadratic eq^n in m. known as the indicial eq^n. which gives two values m_1 and m_2 of m.

Step-5:

Form of solutions:-

The indicial eq^n obtained in step-4 gives two values of m which may be

1) Distinct and donot differ by an integer.

2) Equal

3) Distinct and differ by an integer.

Step-6: Next, on equating to zero the coefficients of the other powers of x, find the values of $a_1, a_2, a_3, \dots$ in terms of a_0 or equate to zero the coefficient of the general power of x namely x^{m+n-1} (or) x^{m+n}. Put $n=1,2,3,\dots$ to get the values of $a_1, a_2, a_3, \dots$ in terms of a_0.

Step-7: After getting various coefficients with help of step 5 above, solution is obtained by substituting these in eq^n (2).

working rule for general solution:

Depending on the nature of the roots m_1, m_2 of indicial eq^n we get three cases

Type 1 on Frobenius method:

Roots of indicial eq^n are unequal and donot differ by an integer

Let m_1 and m_2 be the roots of indicial eq^n if m_1, m_2 are distinct and donot differ by an integer then the general solution y_1 and y_2 are obtained by putting $m=m_1$ and $m=m_2$ in the series of y.

y_1, y_2 are two independent solutions.

$\therefore$ The general sol^n is $y = c_1y_1 + c_2y_2$ where c_1, c_2 are arbitrary constants.

Problems:

1) solve in series the eq^n $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$.

Sol^n: Given differential eq^n is $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ —— (1)

comparing eq^n (1) with $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$

where $P(x) = 4x$, $Q(x) = 2$, $R(x) = 1$

$P(x) = 4x = 0$, when $x = 0$.

$\therefore x = 0$ is singular point.

Divide eq^n (1) with $4x$

(1) $\Rightarrow$ $\frac{d^2y}{dx^2} + \frac{2}{4x}\frac{dy}{dx} + \frac{1}{4x}y = 0$

$\Rightarrow$ $\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} + \frac{1}{4x}y = 0$ —— (2)

comparing eq^n ② with $\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$

$P_1(x) = \frac{1}{2x}$ & $P_2(x) = \frac{1}{4x}$

Now $xP_1(x) = x\left(\frac{1}{2x}\right) = \frac{1}{2}$

and $x^2P_2(x) = x^2\left(\frac{1}{4x}\right) = \frac{x}{4}$

$xP_1(x)$ and $x^2P_2(x)$ are analytic at $x=0$

$\therefore$ $x=0$ is a regular singular point.

Let the series sol^n of eq^n ① be of the form

$y = x^m(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots)$

$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots + a_nx^{m+n}$ —— ③

$\frac{dy}{dx} = m \cdot a_0 x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots + (m+n)a_nx^{m+n-1} + \dots$ —— ④

$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + a_1 \cdot m(m+1)x^{m-1} + a_2 \cdot (m+2)(m+1)x^m + a_3 \cdot (m+3)(m+2)x^{m+1} + \dots + a_n(m+n)(m+n-1)x^{m+n-2} + \dots$ —— ⑤

Substituting eq^{n}s ③, ④ and ⑤ in eq^n ①

① $\Rightarrow 4x\left(a_0m(m-1)x^{m-2} + a_1(m+1)(m)x^{m-1} + a_2 \cdot (m+2)(m+1)x^m + a_3 (m+3)(m+2)x^{m+1} + \dots + a_n(m+n)(m+n-1)x^{m+n-2} + \dots\right) + 2(ma_0x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + \dots + a_n(m+n)x^{m+n-1} + \dots) + (a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \dots + a_nx^{m+n} + \dots) = 0$

$\Rightarrow (4xa_0m(m-1)x^{m-2} + 4xa_1(m+1)(m)x^{m-1} + 4xa_2(m+2)(m+1)x^m + 4xa_3(m+3)(m+2)x^{m+1} + \dots + 4xa_n(m+n)(m+n-1)x^{m+n-2} + \dots) + (2ma_0x^{m-1} + 2a_1(m+1)x^m + 2a_2(m+2)x^{m+1} + 2a_3(m+3)x^{m+2} + \dots + 2a_n(m+n)x^{m+n-1} + \dots) + (a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \dots + a_nx^{m+n} + \dots) =$

$\Rightarrow (4a_0m(m-1)+2a_0m)x^{m-1} + (4a_1m(m+1)+2a_1(m+1)+a_0)x^m + (4a_2(m+2)(m+1)+2a_2(m+2)+a_1)x^{m+1} + \cdots + (4a_n(m+n)(m+n-1)+2a_n(m+n)+a_{n-1})x^{m+n-1} + \cdots = 0$ — (6)

Equating to zero the coefficient of the lowest power x, namely x^{m-1}, eq^n (6) gives the indicial eq^n.

$\Rightarrow 4a_0m(m-1)+2a_0m = 0$

$2a_0m(2m-2)+2a_0m = 0$

$2a_0m[2m-2+1] = 0$

$2a_0m(2m-1) = 0$

$\Rightarrow m(2m-1) = 0$

$m = 0, \ m = \frac{1}{2}$

The indicial eq^n roots are $0, \frac{1}{2}$, which are distinct and donot differ by an integer.

The coefficient of x^m in eq^n (6) $= 0$, which gives

$4a_1m(m+1)+2a_1(m+1)+a_0 = 0$

$2a_1(m+1)[2m+1]+a_0 = 0$

$2a_1(m+1)(2m+1) = -a_0$

$$a_1 = \frac{-a_0}{2(m+1)(2m+1)} \quad \text{— (7)}$$

For recurrence relation, equating to zero the coefficient of x^{m+n-1} in eq^n (6), which gives.

$4a_n(m+n)(m+n-1)+2a_n(m+n)+a_{n-1} = 0$

$$2a_n(m+n)\left[2(m+n-n+1)\right]+a_{n-1}=0$$

$$2a_n(m+n)\left[2m+2n-1\right]+a_{n-1}=0$$

$$2a_n(m+n)(2m+2n-1)=-a_{n-1}$$

$$\therefore \boxed{a_n=\frac{-a_{n-1}}{2(m+n)(2m+2n-1)}} \quad \text{—(8)}$$

case(i):

Let $m=0$, then

$$(8) \Rightarrow \quad a_n=\frac{-(a_{n-1})}{2(0+n)(0+2n-1)}$$

$$a_n=\frac{-(a_{n-1})}{2n(2n-1)} \quad \text{—(9)}$$

Putting $n=1,2,3,\ldots$ in eq^n (9), we have

$$n=1,\quad a_1=\frac{-a_0}{2(1)}=-\frac{a_0}{2}$$

$$\boxed{a_1=\frac{-a_0}{2!}}$$

$$n=2,\quad a_2=\frac{-a_1}{4(3)}=\frac{-a_1}{4\cdot 3}=-\left(-\frac{a_0}{2!}\right)\left(\frac{a_1}{4\cdot 3}\right)$$

$$\boxed{a_2=\frac{a_0}{4!}}$$

$$n=3,\quad a_3=\frac{-a_2}{6(5)}=\frac{-1}{30}\left(\frac{a_0}{4!}\right)$$

$$\boxed{a_3=\frac{-a_0}{6!}} \quad \text{and so on}$$

Putting $m=0$ in eq^n (9) and substituting the above value's a_0, we get a particular sol^n of eq^n as

$$y_1 = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots$$

$$y_1 = a_0 + \left(\frac{-a_0}{2!}\right)x + \left(\frac{a_0}{4!}\right)x^2 + \left(\frac{a_0}{6!}\right)x^3 + \cdots$$

$$\boxed{y_1 = a_0\left(1 - \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \cdots\right)} \quad \text{——— (10)}$$

case(ii):

$m = 1/2$, then

(8) $\Rightarrow$ $$a_n = \frac{-a_{n-1}}{2(\frac{1}{2}+n)(2(\frac{1}{2})+2n-1)}$$

$$= \frac{-a_{n-1}}{2\left(\frac{1+2n}{2}\right)(2n)}$$

$$a_n = \frac{-a_{n-1}}{2n(2n+1)} \quad \text{——— (11)}$$

Putting $n=1,2,3,\cdots$ in eq^n (11)

$n=1$, $a_1 = \dfrac{-a_0}{2(3)} = \dfrac{-a_0}{3!}$

$n=2$, $a_2 = \dfrac{-a_1}{4(5)} = \dfrac{-a_1}{20} = \dfrac{-1}{20}\left(-\dfrac{a_0}{3!}\right)$

$a_2 = \dfrac{a_0}{5!}$

$n=3$, $a_3 = \dfrac{-a_2}{6(7)} = \dfrac{-1}{42}\left(\dfrac{a_0}{5!}\right)$

$a_3 = \dfrac{-a_0}{7!}$ and so on

Putting $m = 1/2$ in eq^n ③ and substituting the above values.

we get a particular sol^n of eq^n as

$$y_2 = a_0 x^{1/2} + a_1 x^{1/2+1} + a_2 x^{1/2+2} + a_3 x^{1/2+3} + \cdots$$

$$= x^{1/2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)$$

$$= x^{1/2}\left(a_0 + x\left(\frac{-a_0}{3!}\right) + x^2\left(\frac{a_0}{5!}\right) + x^3\left(\frac{-a_0}{7!}\right) + \cdots\right)$$

$$y_2 = a_0 x^{1/2}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots\right) \text{ — } (12)$$

Thus the complete sol^n of eq^n ① is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 a_0\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right) + c_2 a_0 x^{1/2}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots\right)$$

$$y = A\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots\right) + Bx^{1/2}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots\right)$$

$$y = A\cos\sqrt{x} + B\sin\sqrt{x}$$

where A, B are arbitrary constants.

2) Solve in series the eq^n $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y$ about $x = 0$.

Sol^n: Given differential eq^n is $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$ — ①

comparing eq^n ① with $p(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$

where $p(x) = 9x(1-x)$, $Q(x) = -12$, $R(x) = 4$

$P(x) = 9x(1-x) = 0$, when $x = 0$

$x = 0$, $x = 1$

$\therefore x = 0$ is singular point

Divide eq^n ① with $9x(1-x)$

$$① \Rightarrow \frac{d^2y}{dx^2} - \frac{12}{9x(1-x)}\frac{dy}{dx} + \frac{4y}{9x(1-x)} = 0.$$

$$\frac{d^2y}{dx^2} - \frac{4}{3x(1-x)}\frac{dy}{dx} + \frac{4y}{9x(1-x)} = 0 \quad \text{---} (2)$$

comparing eq^n (2) with $\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$

$$P_1(x) = \frac{-4}{3x(1-x)}, \quad P_2(x) = \frac{4}{9x(1-x)}$$

Now $xP_1(x) = x\left(\frac{-4}{3x(1-x)}\right) = \frac{-4}{3(1-x)}$

$$x^2P_2(x) = x^2\frac{4}{9x(1-x)} = \frac{4x}{9(1-x)}$$

$xP_1(x)$ and $x^2P_2(x)$ are analytic at $x=0$

$\therefore x=0$ is a regular singular point.

Let the series sol^n of eq^n (1) be of the form

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots)$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \dots + a_nx^{m+n} + \dots \quad \text{---} (3)$$

$$\frac{dy}{dx} = m \cdot a_0x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + \dots + a_n(m+n)x^{m+n-1} + \dots \quad \text{---} (4)$$

$$\frac{d^2y}{dx^2} = a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + a_3(m+3)(m+2)x^{m+1} + \dots + a_n(m+n)(m+n-1)x^{m+n-2} + \dots \quad \text{---} (5)$$

substituting eq^ns (3), (4), (5) in eq^n (1)

$$(1) \Rightarrow 9x(1-x)\left[a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + a_3(m+3)(m+2)x^{m+1} + \dots + a_n(m+n)(m+n-1)x^{m+n-2} + \dots\right] - 12\left[m \cdot a_0x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + \dots + a_n(m+n)x^{m+n-1} + \dots\right] + 4\left[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \dots + a_nx^{m+n} + \dots\right] = 0$$

$$\Rightarrow \Big[9a_0 m(m-1)x^{m-1} - 9a_0 m(m-1)x^{m} + 9a_1(m+1)m x^{m} - 9a_1(m+1)m x^{m+1} + 9a_2(m+2)(m+1)x^{m+1} - 9a_2(m+2)(m+1)x^{m+2} + 9a_3(m+3)(m+2)x^{m+2} - 9a_3(m+3)(m+2)x^{m+3} + \cdots + 9a_n(m+n)(m+n-1)x^{m+n-1} - 9a_n(m+n)(m+n-1)x^{m+n} + \cdots\Big] + \Big[-12ma_0x^{m-1} - 12a_1(m+1)x^{m} - 12a_2(m+2)x^{m+1} - 12a_3(m+3)x^{m+2} - \cdots - 12a_n(m+n)x^{m+n-1} - \cdots\Big] + 4a_0x^{m} + 4a_1x^{m+1} + 4a_2x^{m+2} + 4a_3x^{m+3} + \cdots + 4a_nx^{m+n} + \cdots = 0$$

$$\Rightarrow (9a_0m(m-1) - 12a_0m)x^{m-1} + [-9a_0m(m-1) + 9a_1(m+1)m - 12a_1(m+1) + 4a_0]x^{m} + [-9a_1(m+1)m + 9a_2(m+2)(m+1) - 12a_2(m+2) + 4a_1]x^{m+1} + \cdots + [-9a_{n-1}(m+n-1)(m+n-2) + 9a_n(m+n)(m+n-1) - 12a_n(m+n) + 4a_{n-1}]x^{m+n-1} + \cdots = 0 \quad \text{---} (6)$$

Equating to zero, the coefficient of the lowest power namely x^{m-1}, eq^n (6) gives the indicial eq^n

$$\Rightarrow 9a_0m(m-1) - 12a_0m = 0$$

$$3a_0m(3(m-1)-4) = 0$$

$$3a_0m(3m-3-4) = 0$$

$$3a_0m(3m-7) = 0$$

$$m(3m-7) = 0$$

$$m = 0, \quad m = \frac{7}{3}$$

The indicial eq^n roots are $0, 7/3$, which are distinct and donot differ by an integer

The coefficient of x^m in eq^n (6) $= 0$, which gives

$$-9a_0m(m-1)+9a_1(m+1)m-12a_1(m+1)+4a_0=0$$

$$9a_1(m+1)m-12a_1(m+1)=9a_0m(m-1)-4a_0$$

$$3a_1(m+1)[3m-4]=9a_0m(m-1)-4a_0$$

$$a_1=\frac{a_0[9m(m-1)-4]}{(3m+3)(3m-4)}$$

$$a_1=\frac{a_0[9m^2-9m-4]}{(3m+3)(3m-4)} \quad \text{--- (7)}$$

For recurrence relation, equating to zero the coefficient of x^{m+n-1} in eq^n (6), which gives.

$$\Rightarrow \quad -9a_{n-1}(m+n-1)(m+n-2)+9a_n(m+n)(m+n-1)-12a_n(m+n)+4a_{n-1}=0$$

$$9a_n(m+n)(m+n-1)-12a_n(m+n)=9a_{n-1}(m+n-1)(m+n-2)-4a_{n-1}$$

$$3a_n(m+n)[3(m+n-1)-4]=9a_{n-1}(m+n-1)(m+n-2)-4a_{n-1}$$

$$3a_n(m+n)(3m+3n-7)=a_{n-1}[9(m+n-1)(m+n-2)-4]$$

$$\boxed{a_n=\frac{a_{n-1}[9(m+n-1)(m+n-2)-4]}{3(m+n)(3m+3n-7)}} \quad \text{--- (8)}$$

case(i):

Let $m=0$, then

$$(8) \Rightarrow \quad a_n=\frac{a_{n-1}[9(n-1)(n-2)-4]}{3(n)(3n-7)} \quad \text{--- (9)}$$

Putting $n=1,2,3,\dots$ in eq^n (9)

$n=1, \quad a_1 = \dfrac{a_0[9(0)-4]}{3(1)(-4)} = \dfrac{\cancel{+}4a_0}{\cancel{+}12}$

$$\boxed{a_1 = \frac{a_0}{3}}$$

$n=2, \quad a_2 = \dfrac{a_1[9(1)(0)-4]}{3(2)(-1)} = \dfrac{\cancel{+}4a_1}{\cancel{+}6} = \frac{2}{3}\left(\frac{a_0}{3}\right)$

$$\boxed{a_2 = \frac{2a_0}{9}}$$

$n=3, \quad a_3 = \dfrac{a_2[9(2)(1)-4]}{3(3)(2)} = \dfrac{a_2[\cancel{14}^{7}]}{\cancel{18}9} = \frac{7}{9}\left[\frac{2a_0}{9}\right]$

$$\boxed{a_3 = \frac{14a_0}{81}}$$ and so on

Putting $m=0$ in eq^n ③ and substituting the above value's is, we get a particular sol^n of eq^n as

$$y_1 = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$y_1 = a_0 + \frac{a_0}{3}x + \frac{2a_0}{9}x^2 + \frac{14a_0}{81}x^3 + \dots$$

$$\boxed{y_1 = a_0\left[1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \dots\right]} \quad \text{—— (10)}$$

case(ii):

$m = -7/3$, then

$$(8) \Rightarrow a_n = \frac{a_{n-1}\left[9\left(\frac{7}{3}+n-1\right)\left(\frac{7}{3}+n-2\right)-4\right]}{3\left(\frac{7}{3}+n\right)\left(\cancel{3}\left(\frac{7}{\cancel{3}}\right)+3n-\cancel{7}\right)}$$

$$= \frac{a_{n-1}\left[9\left(\frac{4}{3}+n\right)\left(\frac{1}{3}+n\right)-4\right]}{\cancel{3}\left(\frac{7+3n}{\cancel{3}}\right)(3n)}$$

$$a_n = \frac{a_{n-1}\left[\cancel{9}\left(\frac{4+3n}{\cancel{3}}\right)\left(\frac{1+3n}{\cancel{3}}\right)-4\right]}{3n(-7+3n)}$$

$$a_n = \frac{a_{n-1}\left[(4+3n)(1+3n)-4\right]}{3n(-7+3n)} \quad \text{--- (10)}$$

Putting $n=1,2,3,\ldots$ in eq^n (10)

$$n=1, \quad a_1 = \frac{a_0\left[(7(4)-4)\right]}{3(10)} = \frac{a_0(\cancel{24})}{\cancel{30}}$$

$$\boxed{a_1 = \frac{4a_0}{5}}$$

$$n=2, \quad a_2 = \frac{a_1\left[(10)(7)-4\right]}{6(13)} = \frac{a_1[\cancel{66}]}{\cancel{6}(13)} = \frac{11}{13}\left(\frac{4a_0}{5}\right)$$

$$\boxed{a_2 = \frac{44a_0}{65}}$$

$$n=3, \quad a_3 = \frac{a_2\left[(4+9)(1+9)-4\right]}{9(16)} = \frac{a_2\left[130-4\right]}{9\times16} = \frac{a_2[126]}{9\times16} = \frac{7}{8}a_2$$

$$\boxed{a_3 = \frac{7}{8}\left(\frac{44}{65}a_0\right] = \frac{77}{130}a_0}$$

Putting $m=-7/3$ in eq^n (3) and sub the above values in, we get a particular sol^n of eq^n as

$$y_2 = a_0x^{-7/3} + a_1x^{-7/3+1} + a_2x^{-7/3+2} + a_3x^{-7/3+3} + \ldots$$

$$y_2 = x^{-7/3}\left[a_0 + \frac{4a_0}{5}x + \frac{44a_0}{65}x^2 + \frac{77}{130}a_0x^3 + \ldots\right] \quad \text{--- (13)}$$

$$y_2 = x^{-7/3}a_0\left[1+\frac{4}{5}x + \frac{44}{65}x^2 + \frac{77}{130}x^3 + \ldots\right] \quad \text{--- (12)}$$

Thus the complete solⁿ of eqⁿ ① is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 a_0 \left[1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \cdots\right] + c_2 a_0 x^{-7/3}\left[1 + \frac{4}{5}x + \frac{44}{65}x^2 + \frac{77}{130}x^3 + \cdots\right]$$

$$y = A\left[1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \cdots\right] + Bx^{-7/3}\left[1 + \frac{4}{5}x + \frac{44}{65}x^2 + \frac{77}{130}x^3 + \cdots\right]$$

where A, B are arbitary constants.

Type 2 on frobenius method:-

If the indicial eqⁿ has two equal roots $m_1 = m_2$, we obtain two linearly independent solution by substituting this value of m_1 in series y and $\frac{\partial y}{\partial m}$. Therefore the complete solution is

$$y = c_1 y_1 + c_2 \left(\frac{\partial y}{\partial m}\right)_{at\, m = m_1}$$

Problems:-

1) solve in series the equation $xy'' + y' + xy = 0$.

solⁿ: Given differential eqⁿ is $xy'' + y' + xy = 0$ —— ①

comparing eqⁿ ① with $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$

where $P(x) = x$, $Q(x) = 1$, $R(x) = x$.

At $x = 0$, $P(x) = x = 0$

$\therefore x = 0$ is a singular point.

Divide eqⁿ ① by x.

$$① \Rightarrow y'' + \frac{1}{x}y' + y = 0 \quad \text{—— ②}$$

comparing eqⁿ ② with $y'' + P_1(x)y' + P_2(x)y = 0$

$$P_1(x) = \frac{1}{x}, \quad P_2(x) = 1$$

Now, $xP_1(x) = x\left(\frac{1}{x}\right) = 1$

Thus the complete solⁿ of eqⁿ ① is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 a_0 \left[1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \cdots\right] + c_2 a_0 x^{-7/3}\left[1 + \frac{4}{5}x + \frac{44}{65}x^2 + \frac{77}{130}x^3 + \cdots\right]$$

$$y = A\left[1 + \frac{x}{3} + \frac{2x^2}{9} + \frac{14x^3}{81} + \cdots\right] + Bx^{-7/3}\left[1 + \frac{4}{5}x + \frac{44}{65}x^2 + \frac{77}{130}x^3 + \cdots\right]$$

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Divide eqⁿ ① by x.

① ⇒ $y'' + \frac{1}{x}y' + y = 0$ —— ②

comparing eqⁿ ② with $y'' + P_1(x)y' + P_2(x)y = 0$.

$P_1(x) = \frac{1}{x}$, $P_2(x) = 1$.

Now, $xP_1(x) = x\left(\frac{1}{x}\right) = 1$

$x^2 P_2(x) = x^2(1) = x^2$

$xP_1(x)$ and $x^2P_2(x)$ are analytic at $x=0$

$\therefore$ $x=0$ is a regular singular point.

Let the series sol^n of eq^n be of the form.

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots)$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \cdots + a_nx^{m+n} + \cdots \quad —③$$

$$\frac{dy}{dx} = a_0 \cdot mx^{m-1} + a_1 \cdot (m+1)x^m + a_2 \cdot (m+2)x^{m+1} + a_3 \cdot (m+3)x^{m+2} + \cdots + a_n(m+n)x^{m+n-1} + \cdots \quad —④$$

$$\frac{d^2y}{dx^2} = a_0 \cdot m(m-1)x^{m-2} + a_1 \cdot (m+1)mx^{m-1} + a_2 \cdot (m+2)(m+1)x^m + \cdots + a_n(m+n)(m+n-1)x^{m+n-2} + \cdots \quad —⑤$$

Now sub eq^{n}s ③, ④ & ⑤ in eq^n ①

$$① \Rightarrow x\left[a_0 \cdot m(m-1)x^{m-2} + a_1 \cdot (m+1)mx^{m-1} + a_2 \cdot (m+2)(m+1)x^m + \cdots + a_n(m+n)(m+n-1)x^{m+n-2} + \cdots\right] + \left[a_0mx^{m-1} + a_1 \cdot (m+1)x^m + a_2 \cdot (m+2)x^{m+1} + \cdots + a_n(m+n)x^{m+n-1} + \cdots\right] + x\left[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots + a_nx^{m+n} + \cdots\right] = 0$$

$$\Rightarrow \left[a_0m(m-1)x^{m-1} + a_1(m+1)mx^m + a_2(m+2)(m+1)x^{m+1} + \cdots + a_n(m+n)(m+n-1)x^{m+n-1} + \cdots\right] + \left(a_0mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots + a_n(m+n)x^{m+n-1}\right] + \left[a_0x^{m+1} + a_1x^{m+2} + a_2x^{m+3} + \cdots + a_nx^{m+n+1} + \cdots\right] = 0$$

Equate the lowest degree of x to zero i.e., $x^{m-1} = 0$ —⑥

$\Rightarrow a_0m(m-1) + a_0m = 0$

$a_0[m^2 - \cancel{m} + \cancel{m}] = 0$

$a_0m^2 = 0$

$m^2 = 0$

$m = 0, 0.$

$\therefore$ the indicial eq^n roots are equal.

Now the coefficient of x^m in eq^n (6)

$a_1 m(m+1) + a_1(m+1) = 0$

$a_1(m+1)[m+1] = 0$

$a_1(m+1)^2 = 0$

$\Rightarrow a_1 = 0 \quad [\because m = 0]$

case(1): $m = 0$:-

Now, the coefficient of x^{m+n} in eq^n (6), which gives.

$$\left((a_{n+1})(m+n+1)(m+n) + a_{n+1}(m+n+1) + \cdots + a_{n-1}\right) = 0$$

$$\Rightarrow (a_{n+1})(m+n+1)(m+n+1) = -a_{n-1}$$

$$\boxed{a_{n+1} = \frac{-a_{n-1}}{(m+n+1)^2}} \quad \text{—— (7)}$$

when $m=0$, $a_{n+1} = \dfrac{-a_{n-1}}{(n+1)^2}$

i.e., $a_n = \dfrac{-a_{n-2}}{n^2}$ —— (8), $n \geq 2$

$n = 2 \Rightarrow a_2 = -\dfrac{a_0}{4}$

$n = 3$, $a_3 = \dfrac{-a_1}{9} = 0 \quad [\because a_1 = 0]$

$a_3 = 0$

$n = 4$, $a_4 = \dfrac{-a_2}{16} = \dfrac{-1}{16}\left[\dfrac{-a_0}{4}\right]$

$a_4 = \dfrac{a_0}{64}$

$n = 5$, $a_5 = \dfrac{-a_3}{25} = 0$

$a_5 = 0$

$n=6, \quad a_6 = -\frac{a_4}{36} = -\frac{1}{36}\left(\frac{a_0}{64}\right)$

$$a_6 = \frac{-a_0}{2304}$$

Now sub the values $a_1=0,\ a_2 = \frac{-a_0}{(m+2)^2},\ a_3=0,\ a_4 = \frac{-a_0}{(m+2)^2(m+4)^2}$

$a_5=0,\ a_6 = \frac{-a_0}{(m+2)^2(m+4)^2(m+6)^2}$ ---- in eq^n (3)

$$(3) \Rightarrow y = x^m a_0 + 0 - \frac{a_0}{(m+2)^2} x^{m+2} + 0 + \frac{a_0}{(m+2)^2(m+4)^2} \cdot x^{m+4} + 0 - \frac{a_0}{(m+2)^2(m+4)^2(m+6)^2} + \cdots$$

$$y = a_0 x^m \left(1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \cdots\right) \quad -(9)$$

which is the sol^n when $m=0$ it gives only one solution.

and the second solution is given by $\frac{\partial y}{\partial m}$ when $m=m_1=0$

$\therefore$ The first sol^n of d.e is obtained by substituting $m=0$ in eq^n (9)

$$\therefore \boxed{y_1 = a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right)}$$

case(ii): $\frac{\partial y}{\partial m}$ at $m=0$.

Now differentiating eq^n (9) partially with respect m

$$\frac{\partial y}{\partial m} = a_0 x^m \log_e x \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots\right) + a_0 x^m \frac{\partial}{\partial m}\left(1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \cdots\right) - (10)$$

$\left[\because \frac{d}{dx}(a^x) = a^x \log_e a\right]$

$$\frac{\partial}{\partial m}(1) = 0$$

$$\frac{\partial}{\partial m}\left(\frac{-x^2}{(m+2)^2}\right) = (-x^2)\left(\frac{-2}{(m+2)^3}\right) = \frac{2x^2}{(m+2)^3}$$

$$\frac{\partial}{\partial m}\left(\frac{x^4}{(m+2)^2(m+4)^2}\right) = x^4\left[\left(\frac{-2}{(m+2)^3}\right)\left(\frac{1}{(m+4)^2}\right) + \frac{1}{(m+2)^2}\cdot\left(\frac{-2}{(m+4)^3}\right)\right]$$

$$= -2x^4\left[\frac{1}{(m+2)^3(m+4)^2} + \frac{1}{(m+2)^2(m+4)^3}\right]$$

$$= \frac{-2x^4}{(m+2)^2(m+4)^2}\left[\frac{1}{m+2} + \frac{1}{m+4}\right]$$

(10) $\Rightarrow$ $$\frac{\partial y}{\partial m} = \left(a_0 x^m \log x\left(1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2}\right.\right.$$

$$\left. + \cdots\right) + a_0 x^m \left(0 + \frac{2x^2}{(m+2)^3} - \frac{2x^4}{(m+2)^2(m+4)^2}\left(\frac{1}{m+2} + \frac{1}{m+4}\right)\right.$$

$$\left. + \cdots\cdots\right)$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{at\, m=0} = a_0 x^0 \log x\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) +$$

$$a_0 x^0\left(\frac{2x^2}{2^3} - \frac{2x^4}{2^2\cdot 4^2}\left(\frac{1}{2} + \frac{1}{4}\right) + \cdots\right)$$

$$y_2 = a_0^{\log x}\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) + a_0\left(\frac{x^2}{2^2} - \frac{x^4}{2^2\cdot 4^2}\cdot\left(\frac{3}{4}\right) + \cdots\right)$$

$$\therefore y_2 = y_1 \log x + a_0 x^m\left(\frac{x^2}{2^2} - \frac{1}{2^2\cdot 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2\cdot 4^2\cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 \cdots\right)$$

$$y_2 = y_1 \log x + a_0\left(\frac{x^2}{2^2} - \frac{1}{2^2\cdot 4^2}(1 + 1/2)x^4 + \frac{1}{2^2\cdot 4^2\cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 \cdots\right)$$

Hence the General solⁿ is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) + c_2\left[y_1 \log x + a_0\left(\frac{x^2}{2} - \frac{1}{2^2\cdot 4^2}(1 + 1/2)x^4 + \frac{1}{2^2\cdot 4^2\cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 \cdots\right)\right]$$

Type - 3 on frobenius method :-

Roots of indicial eqⁿ are unequal differ by integer and making a coefficient y infinity.

1) If an indicial eqⁿ has two unequal roots m_1 and m_2, say $m_1 > m_2$ differering by integer and is the some of the coefficients of y become infinity

we modify the form of y by replacing a_0 by $b_0(m-m_2)$ then we obtain two linearly independent solutions by $m=m_2$ in the modified form of y and $\frac{\partial y}{\partial m}$.

2) The result of putting $m=m_1$ in y gives a numerical multiple of that obtained by $m=m_2$ and hence then reject the solution obtained by $m=m_1$ in y. so therefore the complete solⁿ of given differential eqⁿ is

$$y = c_1[y]_{m=m_2} + c_2\left[\frac{\partial y}{\partial m}\right]_{m=m_2}$$

Problems :-

1) obtain the series solⁿ of the eqⁿ $x(1-x)y''-3xy'-y=0$.

Solⁿ: Given differential eqⁿ is $x(1-x)y''-3xy'-y=0$. — (*)

i.e., $(x-x^2)y''-3xy'-y=0$ — ①

$$P(x)y''+Q(x)y'+R(x)y=0$$

$P(x) = x-x^2$, $Q(x) = -3x$, $R(x) = -1$

$P(x)=0$, at $x=0$

$\therefore x=0$ is a singular point.

Dividing eqⁿ① by $(x-x^2)$

$$① \Rightarrow y'' - \frac{3x}{x-x^2}y' - \frac{y}{x-x^2} = 0$$

$$y'' - \frac{3}{1-x} y' - \frac{y}{x-x^2} = 0 \quad — (2)$$

comparing eq^n (1) with $y'' + P_1(x)y' + P_2(x)y = 0$

$$P_1(x) = \frac{-3}{1-x}, \quad P_2(x) = \frac{-1}{x-x^2}$$

$$xP_1(x) = \frac{-3x}{1-x} \quad \text{and} \quad x^2P_2(x) = \frac{-x}{1-x}$$

$xP_1(x)$ and $x^2P_2(x)$ are analytic when $x=0$

$\therefore x=0$ is a regular singular point.

Assume that the sol^n of eq^n (1) be of the form

$$y = x^m[a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots]$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \dots + a_nx^{m+n} + \dots \quad — (3)$$

$$\frac{dy}{dx} = a_0 \cdot mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + a_3(m+3)x^{m+2} + \dots +$$
$$a_n(m+n)x^{m+n-1} + \dots \quad — (4)$$

$$\frac{d^2y}{dx^2} = a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + \dots +$$
$$a_n(m+n)(m+n-1)x^{m+n} + \dots \quad — (5)$$

sub eq^n (3), (4) and (5) in eq^n (1)

$$(1) \Rightarrow (x-x^2)[a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + \dots +$$
$$a_n(m+n)(m+n-1)x^{m+n} + \dots] - 3x[a_0mx^{m-1} + a_1(m+1)x^{m+1} + a_2(m+2)x^{m+1}$$
$$+ \dots + a_n(m+n)x^{m+n-1} + \dots] - [a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots + a_nx^{m+n} + \dots$$
$$= 0$$

$$\Rightarrow [a_0m(m-1)x^{m-1} + a_1m(m+1)x^m + a_2(m+1)(m+2)x^{m+1} + \dots +$$
$$a_n(m+n)(m+n-1)x^{m+n+1} + \dots) - (a_0(m)(m-1)x^m + a_1m(m+1)x^{m+1} +$$
$$a_2(m+1)(m+2)x^{m+2} + \dots + a_n(m+n)(m+n-1)x^{m+n+2} + \dots) -$$

$$\left(3a_0 m x^{m} + 3a_1(m+1)x^{m+1} + 3a_2(m+2)x^{m+2} + \cdots 3a_n(m+n)x^{m+n} + \cdots\right)$$

$$-\left(a_0 x^{m} + a_1 x^{m+1} + a_2 x^{m+2} + \cdots + a_n x^{m+n} + \cdots\right) = 0 \quad \text{——} (6)$$

Equating to zero the coefficient of the lowest power x. namely x^{m-1}, we get

$$\Rightarrow a_0 m(m-1) = 0$$

$$m(m-1) = 0$$

$$m = 0, \; m = 1.$$

The indicial eq^n roots are $m = 0, 1$ which are distinct & differ by an integer.

Equating to zero the coefficient of x^m in eq^n (6), we get

$$a_1 m(m+1) - a_0 (m)(m-1) - 3a_0 m - a_0 = 0$$

$$a_1 m(m+1) - a_0 m^2 + a_0 m - 3a_0 m - a_0 = 0$$

$$a_1 m(m+1) - a_0 m^2 - 2a_0 m - a_0 = 0$$

$$a_1 m(m+1) - a_0 (m^2 + 2m + 1) = 0$$

$$a_1 m(m+1) = a_0 (m+1)^2$$

$$a_1 = \frac{a_0 (m+1)^{\cancel{2}}}{m\cancel{(m+1)}}$$

$$\boxed{a_1 = \frac{m+1}{m} a_0} \quad \text{——} (7)$$

To find the recurrence relation we equate to zero the coefficient of x^{m+n-1}, we get

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$$\left(a_{n-2}(m+n-2)(m+n-3) - a_{n-3}(m+n-3)(m+n-4) - 3a_{n-1}(m+n-1) - a_{n-1}\right) = 0.$$

$$\Rightarrow (m+n-3)\left[a_{n-2}(m+n-2) - a_{n-3}(m+n-4)\right] + a_{n-1}\left[-3m-3n+3-1\right] = 0$$

$$\Rightarrow (m+n-3)\left[a_{n-2}(m+n-2) - a_{n-3}(m+n-4)\right] + a_{n-1}\left[2-3m-3n\right] = 0$$

$$\Rightarrow a_n(m+n)(m+n-1) - a_{n-1}(m+n-2)(m+n-1) + a_{n-1}(2-3m-3n) = 0$$

[∵ By replacing $n=n+2$] in first term].

$$\Rightarrow a_n(m+n)(m+n-1) - a_{n-1}\left[(m+n-1)(m+n-2) - (2-3m-3n)\right] = 0$$

$$\Rightarrow a_n(m+n)(m+n-1) - a_{n-1}\left[m^2+mn-2\cancel{m}+nm+n^2-2\cancel{n}-\cancel{m}-\cancel{n}+\cancel{2}-\cancel{2}+3\cancel{m}-3\cancel{n}\right] = 0$$

$$\Rightarrow a_n(m+n)(m+n-1) - a_{n-1}(m^2+n^2+2mn) = 0$$

$$a_n(m+n)(m+n-1) - a_{n-1}(m+n)^2 = 0$$

$$a_n = \frac{(m+n)^{\cancel{2}}}{\cancel{(m+n)}(m+n-1)}\, a_{n-1}$$

$$\boxed{a_n = \frac{(m+n)}{(m+n-1)}\, a_{n-1}} \quad \text{— (8)}, \quad n \geq 1$$

putting $n = 1, 2, 3, \ldots$ in eq^n (8), we get

$$n=1, \quad a_1 = \frac{(m+1)}{(m)}\, a_0$$

$$n=2, \quad a_2 = \frac{(m+2)}{(m+1)}\, a_1 = \frac{(m+2)}{\cancel{(m+1)}}\,\frac{\cancel{(m+1)}}{m}\, a_0$$

$$a_2 = \frac{(m+2)}{m}\, a_0.$$

$n=3, \quad a_3 = \frac{m+3}{m} a_0$ and so on

putting these values in eq^n (3) we get

$$(3) \Rightarrow y = a_0 x^m + \frac{(m+1)}{m} a_0 x^{m+1} + \frac{(m+2)}{m} a_0 \cdot x^{m+2} + \frac{(m+3)}{m} a_0 x^{m+3} + \cdots$$

$$y = a_0 \left[x^m + \frac{(m+1)}{m} x^{m+1} + \frac{(m+2)}{m} x^{m+2} + \frac{(m+3)}{m} x^{m+3} + \cdots\right]$$

$$y = a_0 x^m \left[1 + \frac{(m+1)}{m} x + \frac{(m+2)}{m} x^2 + \frac{(m+3)}{m} x^3 + \cdots\right] \quad \text{--- (9)}$$

Hence eq^n (9) is a solution of eq^n (1), if $m=0$ or $m=1$

[If $m=0$ or $m=1$]

But if we put $m=0$ in eq^n (9), we find that due to presence of the factor m in their denominators, the coefficients become infinity.

To overcome this difficulty we put $a_0 = b_0(m-0)$

i.e., $a_0 = b_0 m$ in eq^n (9)

$$(9) \Rightarrow y = b_0 m x^m \left(1 + \frac{m+1}{m} x + \left(\frac{m+2}{m}\right) x^2 + \cdots\right)$$

$$y = b_0 x^m \left(m + (m+1)x + (m+2)x^2 + \cdots\right) \quad \text{--- (10)}$$

If $m=0$, eq^n (10) becomes.

$$(10) \Rightarrow y_1 = b_0 x^0 \left[0 + x + 2x^2 + \cdots\right]$$

$$y_1 = b_0 \left[x + 2x^2 + 3x^3 + \cdots\right] \quad \text{--- (11)}$$

This gives only one solution instead of two and the second solution is given by $\frac{\partial y}{\partial m}$ when $m=0$

$$\therefore y_2 = \left(\frac{\partial y}{\partial m}\right)_{\text{when } m=0}$$

$$y_2 = \frac{\partial}{\partial m}\left[b_0 x^m \{m+(m+1)x+(m+2)x^2+\cdots\}\right]$$

$$= b_0 x^m \log x\,(m+(m+1)x+(m+2)x^2+\cdots] + b_0 x^m\left[1+(1+0)x+(1+0)x^2+\cdots\right]$$

$$y_2 = \log x\left[b_0 x^m(m+(m+1)x+(m+2)x^2+\cdots)\right] + b_0 x^m\left[1+x+x^2+\cdots\right]$$

when $m=0$, $y_2 = \log x\left[b_0(x+2x^2+3x^3+\cdots)\right] + b_0\left[1+x+x^2+\cdots\right]$

$$y_2 = b_0 \log x\,(x+2x^2+3x^3+\cdots) + b_0(1+x+x^2+\cdots) \quad \text{—— (12)}$$

Hence the complete solⁿ of eqⁿ (1) is $y = c_1y_1 + c_2y_2$

$$\therefore y = c_1 b_0(x+2x^2+3x^3+\cdots) + c_2 b_0 \log x\,(x+2x^2+3x^3+\cdots) + c_2 b_0(1+x+x^2+\cdots)$$

$$y = (A + B\log x)(x+2x^2+3x^3+\cdots) + B(1+x+x^2+\cdots)$$

where $A = c_1 b_0$ and $B = c_2 b_0$ are arbitrary constants.

Note:-

For $m=1$ eqⁿ (9) gives $y = a_0 x\left[1+2x+3x^2+4x^3+\cdots\right]$

$$= a_0\left[x+2x^2+3x^3+\cdots\right]$$

which is the same as the eqⁿ (11)

2) Solve $x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$.

Solⁿ: Given differential eqⁿ is $x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$ —— (14)

i.e., $(x-x^2)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$ —— (1)

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

$P(x) = x - x^2 \quad Q(x) = -(1+3x) \quad R(x) = -1$.

$p(x)=0$, at $x=0$

$\therefore x=0$ is a singular point.

Dividing eq^n ① with $(x-x^2)$

$$① \Rightarrow \frac{d^2y}{dx^2} - \frac{(1+3x)}{x-x^2}\frac{dy}{dx} - \frac{1}{x-x^2}y = 0 \quad —②$$

comparing eq^n ② with $\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$

$$P_1(x) = \frac{-(1+3x)}{x-x^2}, \quad P_2(x) = \frac{-1}{x-x^2}$$

$$xP_1(x) = x \times \frac{-(1+3x)}{x-x^2}$$

$$xP_1(x) = \frac{-(1+3x)}{1-x} \quad \text{and} \quad x^2P_2(x) = \frac{-x^2}{x-x^2} = \frac{-x}{1-x}$$

$xP_1(x)$ and $x^2P_2(x)$ are analytic when $x=0$

$\therefore x=0$ is a regular singular point.

Assume that the sol^n of eq^n ① be of the form

$$y = x^m\left[a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots\right]$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \dots + a_nx^{m+n} + \dots \quad —③$$

$$\frac{dy}{dx} = a_0 \cdot mx^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \dots + a_n(m+n)x^{m+n-1} + \dots \quad —④$$

$$\frac{d^2y}{dx^2} = a_0 \cdot m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + \dots + a_n(m+n)(m+n-1)x^{m+n-2} + \dots \quad —⑤$$

Sub eq^n ③, ④ & ⑤ in eq^n ①

$$(1) \Rightarrow (x-x^2)\left[a_0 m(m-1)x^{m-2} + a_1(m+1)m x^{m-1} + a_2(m+2)(m+1)x^m + \cdots + a_n(m+n)(m+n-1)x^{m+n-2} + \cdots\right] - (1+3x)\left[a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots + a_n(m+n)x^{m+n-1} + \cdots\right] - \left[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots a_n x^{m+n} + \cdots\right] = 0$$

$$\Rightarrow a_0 m(m-1)x^{m-1} + a_1(m+1)m x^m + a_2(m+2)(m+1)x^{m+1} + \cdots + a_n(m+n)(m+n-1)x^{m+n-1} - \left[a_0 m(m-1)x^m + a_1(m+1)m x^{m+1} + a_2(m+2)(m+1)x^{m+2} + \cdots + a_n(m+n)(m+n-1)x^{m+n} + \cdots\right] - \left[a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots + a_n(m+n)x^{m+n-1} + \cdots\right] + 3a_0 m x^m + 3a_1(m+1)x^{m+1} + 3a_2(m+2)x^{m+2} + \cdots + 3a_n(m+n)x^{m+n} + \cdots\Big] - \left[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots + a_nx^{m+n} + \cdots\right] = 0$$

$y = (A + B\log x)(1 + 2x^2 + 2\cdot 3x^3 + 3\cdot 4x^4 + \cdots) + B(-1 + x + 5x^2 + 11x^3 + \cdots)$

$$\Rightarrow (a_0 m(m-1) - a_0 m)x^{m-1} + \left[a_1(m+1)m - a_0 m(m-1) - a_1(m+1) - 3a_0 m - a_0\right]x^m + \cdots + \left[a_n(m+n)(m+n-1) - a_{n-1}(m+n-1)(m+n-2) - a_n(m+n) - 3a_{n-1}(m+n-1) - a_{n-1}\right]x^{m+n-1} = 0 \quad (6)$$

Equating to zero the coefficient of the lowest power x, namely x^{m-1}, we get.

$$\Rightarrow a_0 m(m-1) - a_0 m = 0$$

$$a_0[m(m-1) - m] = 0$$

$$a_0[m^2 - m - m] = 0$$

$$m^2 - 2m = 0$$

$$m(m-2) = 0$$

$$m = 0,\ m = 2$$

The indicial eq^n roots are $m = 0, 2$ which are distinct & differ by an integer.

Equating to zero the coefficient of x^m in eq^n (6), we get

$$a_1(m+1)m - a_0 m(m+1) - a_1(m+1) - 3a_0 m - a_0 = 0$$

$$[a_1(m+1)][m-1] - a_0[m(m+1)+3m+1] = 0$$

$$a_1[m^2 - \cancel{m} + \cancel{m} - 1] - a_0[m^2+m+3m+1] = 0$$

$$a_1[m^2-1] - a_0[m^2+4m+1] = 0$$

$$a_1 = \frac{a_0[m^2+4m+1]}{m^2-1} \quad \text{—} \cancel{(6)}$$

$$\cancel{a_1 = \frac{a_0[m-1]^{2}}{(m+1)(m-1)}}$$

$$\cancel{a_1 = \frac{a_0[m-1]}{(m+1)} \quad \text{—} (7)}$$

$$a_1 = \frac{a_0[m^2+4m+1]}{m^2-1} \quad \text{—} (7)$$

To find the recurrence relation we equate to zero the coefficient of x^{m+n-1}, we get

$$a_n(m+n)(m+n-1) - a_{n-1}(m+n-1)(m+n-2) - a_n(m+n) - 3a_{n-1}(m+n-1) - a_{n-1} = 0$$

$$\Rightarrow a_n(m+n)[m+n-1-1] - a_{n-1}[(m+n-1)(m+n-2)+3(m+n-1)+1] = 0$$

$$\Rightarrow a_n(m+n)(m+n-2) - a_{n-1}[(m+n-1)[m+n-2+3]+1] = 0$$

$$a_n(m+n)(m+n-2) - a_{n-1}[(m+n-1)(m+n+1)+1] = 0$$

$$a_n(m+n)(m+n-2) = a_{n-1}[(m+n-1)(m+n+1)+1]$$

$$a_n = \frac{a_{n-1}[(m+n-1)(m+n+1)+1]}{(m+n)(m+n-2)} \quad \text{—} (8), \; n \geq 1$$

Module - II

Legendre's differential equation :-

The differential equation

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is known as

Legendre's differential equation. Here n is a real number. But in most applications only integral values of n are needed.

And the solution to this differential equation is called the Legendre's function.

General solution of Legendre's equation:-

Consider the differential eqn

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \longrightarrow ①.$$

comparing eqn ① with

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0.$$

$P_0(x) = (1-x^2)$

$\Rightarrow$ at $x = 0$, $P_0(x) = (1-0) \neq 0$.

$\Rightarrow$ $x=0$ is the ordinary point. & $x = \pm 1$ are singular points.

Now we find the series solution of ① at $x = 0$.

Let $y = \sum_{r=0}^{\infty} a_r x^r$ be the solution of (1) $\longrightarrow$ (2)

differentiating (1), w.r.to x

$$y' = \sum_{r=1}^{\infty} r a_r x^{r-1} \longrightarrow (3)$$

differentiating (3), w.r.to r

$$y'' = \sum_{r=2}^{\infty} r(r-1) a_r x^{r-2} \longrightarrow (4)$$

substituting (1), (2), (3), (4) in (1) we get

$$(1-x^2) y'' - 2xy' + n(n+1)y = 0.$$

$$(1-x^2) \sum_{r=2}^{\infty} r(r-1) a_r x^{r-2} - 2x \sum_{r=1}^{\infty} r a_r x^{r-1} + n(n+1) \sum_{r=0}^{\infty} a_r x^r = 0.$$

$$\sum_{r=2}^{\infty} r(r-1) a_r x^{r-2} - \sum_{r=2}^{\infty} r(r-1) a_r x^r - 2 \sum_{r=1}^{\infty} r a_r x^r$$
$$+ n^2 \sum_{r=0}^{\infty} a_r x^r + n \sum_{r=0}^{\infty} a_r x^r = 0.$$

$$\sum_{r=0}^{\infty} (r+2)(r+1) a_{r+2} x^r - \sum_{r=0}^{\infty} r(r-1) a_r x^r - 2 \sum_{r=0}^{\infty} r a_r x^r + n(n+1) \sum_{r=0}^{\infty} a_r x^r = 0.$$

$$\sum_{r=0}^{\infty} (r+2)(r+1) a_{r+2} x^r - \sum_{r=0}^{\infty} r(r-1) a_r x^r - 2 \sum_{r=0}^{\infty} r a_r x^r + n(n+1) \sum_{r=0}^{\infty} a_r x^r = 0$$

$$\sum_{r=0}^{\infty} \left((r+2)(r+1) a_{r+2} - r(r-1) a_r - 2r a_r + n(n+1) a_r\right] x^r = 0.$$

Now we equate the coefficient of x^r to zero,

$$(r+2)(r+1) a_{r+2} - r(r-1) a_r - 2r a_r + n(n+1) a_r = 0.$$

$$(r+2)(r+1) a_{r+2} - a_r \left[r^2 - r + 2r - n(n+1)\right] = 0$$

$$(r+2)(r+1) a_{r+2} - a_r \left[r^2 + r - n(n+1)\right] = 0.$$

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$$a_{r+2} = \frac{r^2+r-n(n+1)}{(r+2)(r+1)} a_r, \quad r \geq 0. \longrightarrow (5)$$

$$r=0, \quad a_2 = \frac{-n(n+1)}{2\cdot 1} a_0 = -\frac{a_0}{2} n(n+1).$$

$$r=1, \quad a_3 = \frac{2-n(n+1)}{3\cdot 2} a_1 = \frac{a_1}{3!}(2-n(n+1))$$

$$r=2, \quad a_4 = \frac{4+2-n(n+1)}{4\cdot 3} a_2 = \frac{6-n(n+1)}{4\cdot 3} a_2$$

$$a_4 = \frac{[6-n(n+1)]}{4\cdot 3} \cdot \left[\frac{-n(n+1)}{2\cdot 1} a_0\right].$$

since $a_3 = \frac{a_1}{3!}[2-n(n+1)] = \frac{a_1}{3!}[2-n^2-n]$

$$= -\frac{a_1}{3!}[n^2+n-2]$$

$\left[\because n^2+n-2 = n^2+2n-n-2 = n(n+2)-1(n+2) = (n+2)(n-1)\right].$

$$\therefore a_3 = -\frac{a_1}{3!}(n-1)(n+2).$$

since $a_4 = \frac{(6-n(n+1))}{4\cdot 3}\left[\frac{-a_0 n(n+1)}{2}\right]$

$$a_4 = -\frac{a_0}{2\cdot 3\cdot 4} n(n+1)[6-n(n+1)]$$

$$a_4 = -\frac{a_0}{4!} n(n+1)[6-n^2-n]$$

$\left[\because n^2+n-6 = n^2+3n-2n-6 = n(n+3)-2(n+3) = (n+3)(n-2)\right]$

$$a_4 = \frac{a_0}{4!} n(n+1)\{n^2+n-6\}.$$

$$a_4 = \frac{a_0}{4!} n(n+1)(n+3)(n-2).$$

substitute all these values in $y = \sum_{r=0}^{\infty} a_r x^r$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$\Rightarrow y = a_0 + a_1 x - \frac{n(n+1)}{2!} a_0 \cdot x^2 - \frac{(n-1)(n+2)}{3!} a_1 \cdot x^3 + \frac{n(n+1)(n+3)(n-2)}{4!} a_0$$

$$+ \cdots$$

$$y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+3)(n-2)}{4!}x^4 - \cdots\right]$$
$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \cdots\right]$$

$\Rightarrow y = a_0 y_1(x) + a_2 y_2(x)$.

where $y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+3)(n-2)}{4!}x^4 - \cdots$

$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \cdots$

$\Rightarrow y = c_1 y_1(x) + c_2 y_2(x)$.

$\therefore$ The series $y_1(x)$ & $y_2(x)$ converge for $|x|<1$.

we can observe that $y_1(x)$ contains even powers of x only and $y_2(x)$ contains odd powers of x.

—— x ——

Here, the two solutions $y_1(x)$ & $y_2(x)$ are linear independent solutions of the Legendre's equation.

As n takes the value zero and even positive integral values, we obtain

$n=0, \quad y_1(x) = 1 \longrightarrow (6)$

$n=2, \quad y(x) = 1-3x^2 \longrightarrow (7)$

$n=4, \quad y_1(x) = 1 - 10x^2 + \frac{35}{3}x^4. \longrightarrow (8)$

This implies that $y_1(x)$ reduces to an even polynomial (polynomial of even powers).

where as $y_2(x)$ is an infinite series.

these polynomials multiplied by (3)

suitable constants are called "Legendre's polynomials".

The Legendre's polynomials are denoted by "$P_n(x)$" where "n" denote the order of the polynomial. Thus, n takes integral values.

One of the linear independent solution of differential eqn (1) is a Legendre polynomial and the second linearly independent solution is an infinite series. The second solution is denoted by "$Q_n(x)$".

In order to explicitly write the expressions for the Legendre polynomial we need to evaluate the multiplicative constants. The values of the multiplicative constants are obtained by setting "$P_n(x)=1$" for $x=1$.

It is to difficult to use the recurrence relation (or) polynomials given in eqns (6) & (7) to multiplicative constants. It is easy to use the "Rodrigue's formula" to find the expression for the Legendre's polynomial. The general solution of Legendre's is $y = a P_n(x) + b Q_n(x)$. where a & b are arbitrary constants.

Legendre's function of first kind $P_n(x)$:- (4).

when n is a +ve integer

and $a_0 = \frac{1.3.5 \text{----} (2n-1)}{n!}$, then we can write

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n . r! (n-2r)! (n-r)!} x^{n-2r}.$$

where $N = \begin{cases} n/2, & \text{if } n \text{ is even.} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$

Leibnitz's formula :-

$$D^n (uv) = uD^n v + nc_1 . Du . D^{n-1} v + nc_2 . D^2u . D^{n-2} v + \cdots\cdots + nc_n D^n u . v].$$

Rodrigue's formula :-

$$P_n(x) = \frac{1}{2^n . n!} . \frac{d^n}{dx^n} (x^2-1)^n.$$

Proof :- Let $v = (x^2-1)^n$.

first we verify that the n^{th} derivative of v i.e, v_n is a solution of Legendre's differential equation

$$(1-x^2) y'' - 2xy' + n(n+1) y = 0. \rightarrow (1).$$

$$v = (x^2-1)^n.$$

differentiating w.r.to x,

$$\frac{dv}{dx} = v_1 = n(x^2-1)^{n-1} . 2x = 2nx(x^2-1)^{n-1}.$$

$$(x^2-1) v_1 = 2nx(x^2-1)^n$$

$$= 2nxv$$

$$\therefore (x^2-1) v_1 = 2nxv \longrightarrow (2)$$

differentiating (2), again w.r.to x

$$(x^2-1) \frac{d^2v}{dx^2} + 2x \frac{dv}{dx} = 2n \left(v + x \frac{dv}{dx}\right)$$

$$(x^2-1) v_2 + 2x v_1 = 2n(v + xv_1).$$

$$(x^2-1) v_2 + 2xv_1 - 2nxv_1 - 2nv = 0$$

$$(x^2-1) v_2 + 2x(1-n) v_1 - 2nv = 0.$$

differentiate n times by applying Leibnitz theorem then for the n^{th} derivative of a product given by

$D^n(uv) = uD^n v + nC_1 Du D^{n-1} v + nC_2 \cdot D^2 u \cdot D^{n-2} v + \cdots + nC_n \cdot D^n u \cdot v$.

$\cancel{D^n[(x^2-1)v_2] + \ldots D^n[xv_1]} =$

$D^n[(x^2-1)v_2] + 2(1-n)D^n[xv_1] - 2n(D^n v) = 0$.

$\left\{(x^2-1)v_{n+2} + n \cdot 2x \cdot v_{n+1} + \frac{n(n-1)}{2} \cdot 2v_n\right\}$

$+ 2(1-n)\{x v_{n+1} + n \cdot 1 \cdot v_n\} - 2n v_n = 0$.

where v_{n+2}, v_{n+1}, v_n are $n^{th}, (n+1)^{th}, (n+2)^{th}$ derivation of v.

$(x^2-1)v_{n+2} + \cancel{2nxv_{n+1}} + (n^2-n)v_n + 2xv_{n+1} - \cancel{2nxv_{n+1}}$

$+ \cancel{2nv_n} - 2n^2 v_n - \cancel{2nv_n} = 0$.

$(x^2-1)v_{n+2} + 2xv_{n+1} - n^2 v_n - nv_n = 0$.

$(x^2-1)v_{n+2} + 2xv_{n+1} - n(n+1)v_n = 0$

$(1-x^2)v_{n+2} - 2xv_{n+1} + n(n+1)v_n = 0$. This can be put in the form $(1-x^2)v_n'' - 2xv_n' + n(n+1)v_n = 0 \rightarrow (3)$

$\Rightarrow v_n$ is a solution of Legendre's equation.

$\therefore$ suppose $P_n(x) = Cv_n$ where c is constant.

$\Rightarrow P_n(x) = C[(x^2-1)^n]_n$ (or) $C D^n[(x^2-1)^n]$

i.e, $P_n(x) = C[(x-1)^n(x+1)^n]_n$.

Applying Leibnitz theorem for the R.H.S we have

$$P_n(x) = c\left[(x-1)^n \cdot D^n (x+1)^n + n \cdot n(x-1)^{n-1} D^{n-1}\left\{(x+1)^n\right\} + \cdots + \left\{D^n (x-1)^n\right\}(x+1)^n\right] \longrightarrow (3)$$

(6)

It should be observed that if

if $z = (x-1)^n$

$z_1 = n(x-1)^{n-1}$

$z_2 = n(n-1)(x-1)^{n-2}$ etc.

$z_n = D^n(x-1)^n = n.(n-1)(n-2)\cdots 2.1\,(x-1)^{n-n}$.

$= n!$

$D^n\left\{(x-1)^n\right\} = n!$

taking $x=1$ in (3) all the terms in R.H.S become zero except the last term which becomes $n!(1+1)^n = n!\,2^n$.

$\therefore P_n(1) = c.\,n!\,2^n$ & $P_n(1) = 1$ by def of $P_n(x)$.

$1 = c.\,n!\,2^n$.

$c = \frac{1}{2^n . n!}$

$\therefore P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n}(x^2-1)^n$.

—— x ——

Binomial theorem :-

$$(x+y)^n = nC_0 \cdot x^n + nC_1 \cdot x^{n-1} \cdot y + nC_2 \cdot x^{n-2} \cdot y^2 + nC_3 \cdot x^{n-3} \cdot y^3$$

$$+ \cdots + nC_r x^{n-r} \cdot y^r + \cdots + nC_n \cdot y^n.$$

	1	3	3	1	
1	4	6	4	1	
1	5	10	10	5	1

Find $P_n(x)$ when $n = 0, 1, 2, \ldots$

we know rodrique's formula

$$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

$\underline{n=0}$:- $P_0(x) = \frac{1}{2^0\, 0!} \cdot 1 = 1$

$\underline{n=1}$:- $P_1(x) = \frac{1}{2 \cdot 1!} \frac{d}{dx} (x^2-1)$

$$= \frac{1}{2} \cdot 2x = x.$$

$$\therefore P_1(x) = x.$$

$\underline{n=2}$:- $P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2} (x^2-1)^2 = \frac{1}{4 \cdot 2} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1)$

$$= \frac{1}{8} \left(\frac{d}{dx} (4x^3 - 4x)\right)$$

$$= \frac{1}{8} (12x^2 - 4) = \frac{1}{2}(3x^2-1)$$

$\underline{n=3}$:- $P_3(x) = \frac{1}{2^3 \cdot 3!} \frac{d^3}{dx^3} (x^2-1)^3$

$$= \frac{1}{8 \cdot 6} \frac{d^3}{dx^3} \left((x^2)^3 - 3(x^2)^2 \cdot 1 + 3(x^2) \cdot 1^2 - 1\right)$$

$$= \frac{1}{8 \cdot 6} \frac{d^3}{dx^3} \left(x^6 - 3x^4 + 3x^2 - 1\right).$$

$$= \frac{1}{48} \frac{d^2}{dx^2} \left(6x^5 - 12x^3 + 6x\right) = \frac{1}{48} \frac{d}{dx} \left(30x^4 - 36x^2 + 6\right)$$

$$= \frac{1}{48} (120x^3 - 72x)$$

$$\therefore P_3(x) = \frac{x}{2}(5x^2-3).$$

$\frac{24}{48}$
$\frac{48}{144}$

|||ly we can find ⑧

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 - 15x).$$

expression of polynomials in terms of $P_n(x)$:—

$$1 = P_0(x)$$

$$x = P_1(x).$$

$$P_2(x) = \frac{3x^2-1}{2} = \frac{3x^2 - P_0(x)}{2}.$$

$$\Rightarrow x^2 = \frac{2P_2(x) + P_0(x)}{3}$$

$$P_3(x) = \frac{5x^3 - 3x}{2} = \frac{5x^3 - 3P_1(x)}{2}$$

$$\Rightarrow 2P_3(x) = 5x^3 - 3P_1(x)$$

$$\Rightarrow 5x^3 = 2P_3(x) + 3P_1(x)$$

$$\Rightarrow x^3 = \frac{2P_3(x) + 3P_1(x)}{5}.$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

$$= \frac{1}{8}\left[35x^4 - 30\left(\frac{2P_2(x) + P_0(x)}{3}\right) + 3P_0(x)\right\}.$$

$$= \frac{1}{8}\left[35x^4 - 10(2P_2(x) + P_0(x)) + 3P_0(x)\right\}.$$

$$= \frac{1}{8}\left[35x^4 - 20P_2(x) - 7P_0(x)\right].$$

$$8P_4(x) = 35x^4 - 20P_2(x) - 7P_0(x).$$

$$35x^4 = 8P_4(x) + 20P_2(x) + 7P_0(x).$$

$$x^4 = \frac{1}{35}\left\{8P_4(x) + 20P_2(x) + 7P_0(x)\right\}.$$

Recurrence Relations (or Formulae)

I. $(2n+1)\, x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$.

proof: we have $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$. — ①

differentiating ① both sides w.r.t. to 't', we have

$$① \Rightarrow \left(-\tfrac{1}{2}\right)(1-2xt+t^2)^{\left(-\frac{1}{2}-1\right)}(-2x+2t) = \sum_{n=0}^{\infty} n.t^{n-1} P_n(x).$$

$$\Rightarrow \left(-\tfrac{1}{\cancel{2}}\right)(1-2xt+t^2)^{-3/2}(-\cancel{2})(x-t) = \sum_{n=0}^{\infty} n.t^{n-1} P_n(x)$$

$$\Rightarrow (1-2xt+t^2)^{-3/2}(x-t) = \sum_{n=0}^{\infty} n.t^{n-1} P_n(x).$$

Multiplying by $(1-2xt+t^2)$ on B.S. of the above Eqⁿ, we get

$$\Rightarrow (1-2xt+t^2)^{-3/2}(x-t)(1-2xt+t^2) = \left[\sum_{n=0}^{\infty} n.t^{n-1} P_n(x)\right](1-2xt+t^2)$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2).\sum_{n=0}^{\infty} n.t^{n-1} P_n(x).$$

$$\Rightarrow (x-t).\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2).\sum_{n=0}^{\infty} n.t^{n-1}. P_n(x).$$

$$\Rightarrow x.\sum_{n=0}^{\infty} t^n P_n(x) - \sum_{n=0}^{\infty} t^{n+1} P_n(x) = \sum_{n=0}^{\infty} n.t^{n-1} P_n(x) - 2x.\sum_{n=0}^{\infty} n t^n P_n(x)$$
$$+ \sum_{n=0}^{\infty} n.t^{n+1} P_n(x).$$

Now Equating the coefficients of t^n on B.S we get

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2xn P_n(x) + (n-1) P_{n-1}(x).$$

$$\Rightarrow 2xn P_n(x) + x P_n(x) = (n+1) P_{n+1}(x) + (n+1-1) P_{n-1}(x).$$

$$\Rightarrow \boxed{(2n+1)\, x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x),} \quad n \geq 1. \longrightarrow ②$$

Note:- Replacing 'n' by $(n-1)$ in this result Eq^n (2), we get

$$(2n-1)xP_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x).$$

(or) $$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x), \quad n \geq 2.$$

II. $$nP_n(x) = xP_n'(x) - P_{n-1}'(x).$$

Proof: we know that $$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{——— (1)}$$

Differentiating Eq^n (1) w.r.t. 't', we get (w.r.t 't')

$$(1) \Rightarrow (-\tfrac{1}{2})(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n.t^{n-1}P_n(x)$$

$$\Rightarrow (1-2xt+t^2)^{-3/2}(x-t) = \sum_{n=0}^{\infty} n.t^{n-1}P_n(x) \longrightarrow (2)$$

Again, differentiating the Eq^n (1) w.r.t 'x', we get

$$(1) \Rightarrow (-\tfrac{1}{2})(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(x)$$

$$\Rightarrow t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} t^n P_n'(x). \longrightarrow (3)$$

Multiplying ~~Eq^n (3)~~ by $(x-t)$ on B.S. ~~of Eq~~ on Eq^n (3).

$$(3) \Rightarrow t(x-t)(1-2xt+t^2)^{-3/2} = (x-t)\sum_{n=0}^{\infty} t^n P_n(x).$$

$$\Rightarrow t.\sum_{n=0}^{\infty} nt^{n-1}P_n(x) = (x-t)\sum_{n=0}^{\infty} t^n P_n'(x) \quad [\because \text{using Eq}^n (2)]$$

$$\Rightarrow \sum_{n=0}^{\infty} n.t^n P_n(x) = x\sum_{n=0}^{\infty} t^n P_n'(x) - t\sum_{n=0}^{\infty} t^n P_n'(x).$$

$$\Rightarrow \sum_{n=0}^{\infty} t^n n P_n(x) = \sum_{n=0}^{\infty} t^n x P_n'(x) - \sum_{n=0}^{\infty} t^{n+1} P_n'(x).$$

Equating the coefficients of t^n on both sides, we get

$$\boxed{n P_n(x) = x P_n'(x) - P_{n-1}'(x)}$$

III. $(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

proof:- From recurrence relation I, we have

$$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x).$$

Differentiating both sides w.r.t. 'x', we get

$$(2n+1)\left[x P_n'(x) + P_n(x)\right] = (n+1) P_{n+1}'(x) + n P_{n-1}'(x).$$

$$\Rightarrow (2n+1) x P_n'(x) + (2n+1) P_n(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x). \quad \text{— ①}$$

From recurrence relation II, we have

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x).$$

$$\Rightarrow x P_n'(x) = n P_n(x) + P_{n-1}'(x) \quad \text{— ②}$$

Now sub. Eq^n ② in Eq^n ①, we get

$$① \Rightarrow (2n+1)\left[n P_n(x) + P_{n-1}'(x)\right] + (2n+1) P_n(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x).$$

$$\Rightarrow (2n+1)\left[n P_n(x)\right] + (2n+1) P_{n-1}'(x) + (2n+1) P_n(x) = (n+1) P_{n+1}'(x) + n P_{n-1}'(x)$$

$$\Rightarrow (2n+1)\left[n P_n(x) + P_n(x)\right] = (n+1) P_{n+1}'(x) + n P_{n-1}'(x) - (2n+1) P_{n-1}'(x)$$

$$\Rightarrow (2n+1)\left[(n+1) P_n(x)\right] = (n+1) P_{n+1}'(x) + (-n-1) P_{n-1}'(x)$$

$$\Rightarrow \cancel{(n+1)}(2n+1) P_n(x) = \cancel{(n+1)}\left[P_{n+1}'(x) - P_{n-1}'(x)\right]$$

$$\therefore (2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \;//.$$

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Note: prove that $P_n' - P_{n-2}' = (2n-1)P_{n-1}(x)$.

Replace n by $(n-1)$ in recurrence relation III, we get

$$(2(n-1)+1)P_{n-1}(x) = P_{n-1+1}'(x) - P_{n-1-1}'(x)$$

$$\Rightarrow (2n-1)P_{n-1}(x) = P_n'(x) - P_{n-2}'(x)$$

$$\therefore P_n' - P_{n-2}' = (2n-1)P_{n-1}$$

IV. $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$

proof:- From recurrence relations II and III, we have

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x) \longrightarrow ①$$

and $(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \longrightarrow ②$

Subtracting Eqn ① from Eqn ②, we get

$$(2n+1-n)P_n(x) = P_{n+1}'(x) - xP_n'(x) - P_{n-1}'(x) - (-P_{n-1}'(x))$$

$$\Rightarrow (n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x) - \cancel{P_{n-1}'(x)} + \cancel{P_{n-1}'(x)}$$

$$\therefore (n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x).$$

V. $(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right]$

proof: From recurrence relations II and IV, we have

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x) \longrightarrow ①$$

and $(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x) \longrightarrow ②$

Replacing n by $(n-1)$ in ②, we get

$$② \Rightarrow (n-1+1)P_{n-1}(x) = P_{n-1+1}'(x) - xP_{n-1}'(x)$$

$\Rightarrow n P_{n-1}(x) = P_n'(x) - x P_{n-1}'(x)$ —— ③

Multiplying eqⁿ ①. by x, we get

① $\Rightarrow n x P_n(x) = x^2 P_n'(x) - x P_{n-1}'(x)$ —— ④

Subtracting eqⁿ ④ from ③, we get.

$n P_{n-1}(x) - n x P_n(x) = P_n'(x) - x P_{n-1}'(x) - (x^2 P_n'(x) - x P_{n-1}'(x))$

$\Rightarrow n P_{n-1}(x) - n x P_n(x) = P_n'(x) - \cancel{x P_{n-1}'(x)} - x^2 P_n'(x) + \cancel{x P_{n-1}'(x)}$

$\Rightarrow n P_{n-1}(x) - n x P_n(x) = (1-x^2) P_n'(x)$.

$\Rightarrow (1-x^2) P_n'(x) = n [P_{n-1}(x) - x P_n(x)]$.

VI. $(1-x^2) P_n'(x) = (n+1) [x P_n(x) - P_{n+1}(x)]$.

proof: From recurrence relation IV, we have

$(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x)$ —— ①.

replacing n by $(n-1)$ in eqⁿ①, we get

① $\Rightarrow (n-1+1) P_{n-1}(x) = P_{n-1+1}'(x) - x P_{n-1}'(x)$

$\Rightarrow n P_{n-1}(x) = P_n'(x) - x P_{n-1}'(x)$ —— ②

Multiplying eqⁿ ②. by x, we get

② $\Rightarrow n x P_{n-1}(x) = x P_n'(x) - x^2 P_{n-1}'(x)$ —— ③

From recurrence relation II, we have.

$n P_n(x) = x P_n'(x) - P_{n-1}'(x)$ —— ④

Subtracting eqⁿ ④ from ③. we get

$n x P_{n-1}(x) - n P_n(x) = \cancel{x P_n'(x)} - x^2 P_{n-1}'(x) - \cancel{x P_n'(x)} + P_{n-1}'(x)$

$$\Rightarrow nxP_{n-1}(x) - nP_n(x) = -x^2 P_{n-1}'(x) + P_{n-1}'(x)$$

$$\Rightarrow n\left[xP_{n-1}(x) - P_n(x)\right] = (1-x^2)P_{n-1}'(x). \quad \text{——} \ (5)$$

Replacing n by $(n+1)$ in Eq^n (5), we get.

$$(5) \Rightarrow (n+1)\left[xP_{n+1-1}(x) - P_{n+1}(x)\right] = (1-x^2)P_{n+1-1}'(x)$$

$$\Rightarrow (n+1)\left[xP_n(x) - P_{n+1}(x)\right] = (1-x^2)P_n'(x)$$

$$\therefore (1-x^2)P_n'(x) = (n+1)\left[xP_n(x) - P_{n+1}(x)\right]$$

Problems related to Legendre's polynomials

1) Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Legendre's polynomials.

Sol:- we have $P_0(x) = 1$, $P_1(x) = x$.

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\Rightarrow 3x^2 - 1 = 2P_2(x)$

$\Rightarrow 3x^2 = 2P_2(x) + 1 = 2P_2(x) + P_0(x)$

$\Rightarrow x^2 = \frac{1}{3}(2P_2(x) + P_0(x))$

$\therefore x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x).$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$\Rightarrow 5x^3 - 3x = 2P_3(x)$

$\Rightarrow 5x^3 = 2P_3(x) + 3x = 2P_3(x) + 3P_1(x)$

$\therefore x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x).$

Substituting the values of x^3, x^2, x and 1, given $f(x)$, we get.

$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] - 5\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] + P_1(x) + 2P_0(x)$

$= \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) - \frac{10}{3}P_2(x) - \frac{5}{3}P_0(x) + P_1(x) + 2P_0(x)$

$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \left(\frac{3}{5} + 1\right)P_1(x) + \left(2 - \frac{5}{3}\right)P_0(x)$

$f(x) = \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$.

2) Express $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

Sol: we have $P_0(x) = 1$, $P_1(x) = x$, $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$.

$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$ and.

$P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$

$\Rightarrow 35x^4 - 30x^2 + 3 = 8P_4(x)$

$\Rightarrow 35x^4 - 30x^2 + 3P_0(x) = 8P_4(x) \quad [\because 1 = P_0(x)]$

$\Rightarrow 35x^4 - 30\left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right) + 3P_0(x) = 8P_4(x)$

$\Rightarrow 35x^4 - 20P_2(x) - 10P_0(x) + 3P_0(x) = 8P_4(x)$

$\Rightarrow 35x^4 = 8P_4(x) + 20P_2(x) + 7P_0(x)$

$\Rightarrow x^4 = \frac{8}{35}P_4(x) + \frac{20}{35}P_2(x) + \frac{7}{35}P_0(x)$

$\therefore x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x).$

Substituting the values of x^4, x^3, x^2, x & 1 in given $f(x)$, we get.

$f(x) = \left(\frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)\right) + 2\left(\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right) + 2\left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right)$

$- P_1(x) - 3P_0(x)$

$= \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x) + \frac{4}{5}P_3(x) + \frac{6}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x) - P_1(x) - 3P_0(x).$

$f(x) = \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{22}{15}P_0(x)$;

3) Show that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ and hence express $2x^2 - 4x + 2$ as a legendre polynomial.

4) Show that $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$ and $x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$.

5) Express the polynomial $f(x) = 4x^3 - 2x^2 - 3x + 8$ in terms of legender polynomials.

6) Express $f(x) = x^3 + 2x^2 - x - 3$ in terms of legendre polynomials.

7) Using Rodrigue's formula, prove that $\int_{-1}^{1} x^m P_n(x)\,dx = 0$, if $m<n$.

Sol: We have Rodrigue's formula that

$$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n}(x^2-1)^n$$

Multiplying the above Eqn by x^m, we get

$$x^m P_n(x) = \frac{1}{2^n . n!} x^m \frac{d^n}{dx^n}(x^2-1)^n$$

Now integrating w.r.t 'x' from the limits -1 to 1 we get

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{1}{2^n n!} \int_{-1}^{1} x^m \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n . n!} \int_{-1}^{1} x^m . d\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]$$

$$= \frac{1}{2^n . n!}\left[\left(x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right)_{-1}^{1} - \int_{-1}^{1} m . x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right]$$

(using integration by Parts)

The first term in the R.H.S becomes zero at both limits because $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ will contain (x^2-1) as a factor.

Hence $$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{(-1)^n m}{2^n . n!} \int_{-1}^{1} x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

Integrating by parts $(n-1)$ times, we get

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{(-1)^n m}{2^n n!} \int_{-1}^{1} (x^2-1)^n . \frac{d^{n-1}}{dx^{n-1}}(x^{m-1})\,dx.$$

$$= 0 \quad \left[\because \frac{d^{n-1}}{dx^{n-1}}(x^{m-1}) = 0 \text{ if } m<n\right].$$

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Generating function for $P_n(x)$:-

The function which generates $P_n(x)$, $n=1,2,3\ldots$ is called the generating function for $P_n(x)$. The generating function for $P_n(x)$ is $(1-2xt+t^2)^{-1/2}$.

Statement:- To show that $P_n(x)$ is the is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$

(or) To show that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$.

(or) prove that $\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x)+P_1(x)t+P_2(x)t^2+\ldots$

proof:- Expanding $(1-2xt+t^2)^{-1/2}$ by Binomial theorem, we have

$$(1-2xt+t^2)^{-1/2} = \left[1-t(2x-t)\right]^{-1/2}$$

$$= 1+\frac{1}{2}t(2x-t)+\left(\frac{1}{2}\cdot\frac{3}{2}\right)\frac{1}{2!}t^2(2x-t)^2+\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\right)\frac{1}{3!}t^3(2x-t)^3+\ldots$$

$$+\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}t^{n-1}(2x-t)^{n-1}+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}t^n(2x-t)^n+\ldots$$

$$\therefore (1-2xt+t^2)^{-1/2} = 1+\frac{1}{2}t(2x-t)+\frac{1.3}{2.4}t^2(2x-t)^2+\frac{1.3.5}{2.4.6}t^3(2x-t)^3+\ldots$$

$$+\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}t^{n-1}(2x-t)^{n-1}+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}t^n(2x-t)^n+\ldots \rightarrow (1).$$

Now coefficient of t^n in

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}t^n(2x-t)^n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}t^n\left[{}^nC_0(2x)^n-{}^nC_1(2x)^{n-1}t^1+\ldots\right]$$

$$= \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots(2n)}(2x)^n$$

$$= \frac{1.3.5\cdots(2n-1)}{2^n(1.2.3\cdots n)}(2x)^n$$

$$= \frac{1.3.5\cdots(2n-1)}{n!}x^n.$$

Now coefficient of t^n in $\frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)}t^{n-1}(2x-t)^{n-1}$

$$= \frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)}t^{n-1}\left[{}^{n-1}c_0(2x)^{n-1} - {}^{n-1}c_1(2x)^{n-2}t^1 + \cdots\right]$$

$$= -\left(\frac{1.3.5\cdots(2n-3)}{2.4.6\cdots(2n-2)}\right)(n-1)\,2^{n-2}.x^{n-2}.t^n$$

$$= -\left(\frac{1.3.5\cdots(2n-3)}{2^{n-1}(1.2.3\cdots(n-1))}\right)\frac{n}{n}.\frac{(2n-1)}{(2n-1)}2^{n-2}.x^{n-2}.(n-1)$$

$$= -\frac{1.3.5\cdots(2n-1)}{\cancel{2^{n-2}}.2(n!)}.\frac{n(n-1)}{(2n-1)}\cancel{2^{n-2}}.x^{n-2}.$$

$$= -\left(\frac{1.3.5\cdots(2n-1)}{n!}\right)\frac{n(n-1)}{2(2n-1)}x^{n-2}.$$

and so on ....

Coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$

$$= \frac{1.3.5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots\right] = P_n(x).$$

$\therefore$ we find that $P_1(x), P_2(x), P_3(x) \ldots$ will be coefficient of $t, t^2, t^3 \ldots$ in the expansion of $(1-2xt+t^2)^{-1/2}$.

$$\therefore (1-2xt+t^2)^{-1/2} = 1 + t P_1(x) + t^2 P_2(x) + \cdots + t^n P_n(x) + \cdots$$

$$= \sum_{n=0}^{\infty} t^n P_n(x)$$

problems:- ① Show that (i) $P_n(1) = 1$, (ii) $P_n(-1) = (-1)^n$.

soln:- we know that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n \rightarrow$ ① say.

(i) putting $x=1$ in eqn ①, we get

$$① \Rightarrow (1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1) \cdot t^n$$

$$\Rightarrow \left((1-t)^2\right)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(1).$$

$$\Rightarrow \sum_{n=0}^{\infty} t^n P_n(1) = (1-t)^{-1} = 1 + t + t^2 + t^3 + \cdots + t^n + \cdots$$

Equating the coefficient of t^n on B.S, we get

$$\boxed{P_n(1) = 1}$$

(ii) putting $x=-1$ in eqn ①, we get

$$① \Rightarrow (1+2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-1) t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(-1) t^n = \left[(1+t)^2\right]^{-1/2}$$

$$= [1+t]^{-1}$$

$$= 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots$$

Equating the coefficient of t^n on B.S. we get

$$\boxed{P_n(-1) = (-1)^n}$$

(2) Using the generating function, prove that

i) $P_{2n+1}(0) = 0$ (ii) $P_{2n}(0) = (-1)^n \cdot \dfrac{(2n)!}{2^{2n}(n!)^2}$

(or) prove that $P_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{(-1)^{n/2}}{2^n\left[\left(\frac{n}{2}\right)!\right]^2}, & \text{if } n \text{ is even} \end{cases}$

Sol:- we know that

$$\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)^{-1/2} \longrightarrow (1) \text{ say.}$$

putting $x=0$, in Eq^n (1), we get

$$(1) \Rightarrow \sum_{n=0}^{\infty} P_n(0)t^n = (1+t^2)^{-1/2}$$

$$= (1-(-t^2))^{-1/2}$$

$$= 1+\frac{1}{2}(-t^2)+\left(\frac{1}{2}\cdot\frac{3}{2}\right)\frac{1}{2!}(-t^2)^2 + \cdots\cdots$$

$$\left[\because (1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\cdots\cdots\right]$$

$$= 1+\frac{1}{2}(-t^2)+\frac{1\cdot 3}{2\cdot 4}(-t^2)^2+\cdots+\frac{1\cdot 3\cdots(2r-1)}{2\cdot 4\cdots(2r)}(-t^2)^r+\cdots$$

$$\therefore \sum_{n=0}^{\infty} P_n(0)\,t^n = \sum_{r=0}^{\infty}(-1)^r\,\frac{1\cdot 3\cdots(2r-1)}{2^r\cdot r!}\,t^{2r} \longrightarrow (2)$$

From (2), it is clear that only even powers of 't' occur in R.H.S. of (2).

(i) Equating the coefficient of t^{2n+1}, i.e., odd powers of 't' on B.S. of Eq^n (2), we get $P_{2n+1}(0)=0$

i.e., $P_n(0) = 0$, if n is odd.

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(ii) Equating the coefficient of t^{2n} on B.S. of (2), we get

$$P_{2n}(0) = (-1)^n \cdot \frac{1\cdot 3 \cdots (2n-1)}{2^n\, n!}$$

$$= (-1)^n \cdot \frac{1\cdot 2\cdot 3\cdot 4 \cdots (2n-1)(2n)}{2^n n! \,[2\cdot 4\cdot 6 \cdots 2n]}$$

$$= (-1)^n \cdot \frac{(2n)!}{2^n \cdot n! \,(2^n \cdot n!)}$$

$$= (-1)^n \cdot \frac{(2n)!}{2^{2n} (n!)^2}$$

Replacing 'n' by $\frac{n}{2}$, we get

$$P_{2(\frac{n}{2})}(0) = (-1)^{n/2} \cdot \frac{(2(\frac{n}{2}))!}{2^{2(n/2)} \cdot (\frac{n}{2}!)^2}$$

$$\Rightarrow P_n(0) = (-1)^{n/2} \frac{n!}{2^n \cdot (\frac{n}{2}!)^2}$$

$$\therefore P_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{(-1)^{n/2}\, n!}{2^n \cdot (\frac{n}{2}!)^2}, & \text{if } n \text{ is even.} \end{cases}$$

(3) prove that $\dfrac{1+z}{z\cdot\sqrt{1-2xz+z^2}} - \dfrac{1}{z} = \sum_{n=0}^{\infty}\left[P_n(x) + P_{n+1}(x)\right]z^n$

Sol^n:- we have the generating function for $P_n(x)$ that

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\cdot z^n \longrightarrow (1) \text{ say}.$$

Now $\dfrac{1+z}{z\cdot\sqrt{1-2xz+z^2}} - \dfrac{1}{z} = \dfrac{1}{z\sqrt{1-2xz+z^2}} + \dfrac{z}{z\cdot\sqrt{1-2xz+z^2}} - \dfrac{1}{z}$

$$= \frac{1}{z}(1-2xz+z^2)^{-1/2} + (1-2xz+z^2)^{-1/2} - \frac{1}{z}$$

$$= \frac{1}{z}\left[\sum_{n=0}^{\infty} P_n(x)z^n\right] + \sum_{n=0}^{\infty} P_n(x)z^n - \frac{1}{z} \quad [\because \text{By Eq}^n (1)]$$

$$= \frac{1}{z}\left[P_0(x) + \sum_{n=1}^{\infty} P_n(x)z^n\right] + \sum_{n=0}^{\infty} P_n(x)z^n - \frac{1}{z}$$

$$= \frac{1}{z}P_0(x) + \frac{1}{z}\sum_{n=1}^{\infty} P_n(x)z^n + \sum_{n=0}^{\infty} P_n(x)z^n - \frac{1}{z}$$

$$= \cancel{\frac{1}{z}} + \frac{1}{z}\sum_{n=1}^{\infty} P_n(x)z^n + \sum_{n=0}^{\infty} P_n(x)z^n - \cancel{\frac{1}{z}} \quad [\because P_0(x)=1]$$

$$= \sum_{n=1}^{\infty} P_n(x)z^{n-1} + \sum_{n=0}^{\infty} P_n(x)z^n$$

$$= \sum_{n=0}^{\infty} P_{n+1}(x)z^n + \sum_{n=0}^{\infty} P_n(x)z^n \quad [\text{Replacing } n \text{ by } n+1 \text{ in the first term}]$$

$$\therefore \frac{1+z}{z\cdot\sqrt{1-2xz+z^2}} - \frac{1}{z} = \sum_{n=0}^{\infty}\left[P_n(x) + P_{n+1}(x)\right]z^n$$

//

4) prove that $P_n(-x) = (-1)^n P_n(x)$ and hence deduce that $P_n(-1) = (-1)^n$.

5) Show that i) $P_n'(1) = \frac{n(n+1)}{2}$ (ii) $P_n'(-1) = (-1)^{n-1} \frac{n(n-1)}{2}$

Orthogonality of Legendre polynomials

Statement:- we shall prove that

$$\int_{-1}^{1} P_m(x) \cdot P_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

(or)

$$\int_{-1}^{1} P_m(x) \cdot P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn}$$

where δ_{mn} is called 'kronecker delta' and is defined by

~~proof:~~

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n. \end{cases}$$

proof:- case (i): when $m \neq n$

we know that legendre's Eqn is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

this Equation can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0 \quad \left[\because \frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) = (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx}\right]$$

$\to$ ① say.

$P_n(x)$ is a solution of legendre Eqn.

$$① \Rightarrow \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n(x) = 0 \quad \text{——} ②$$

$P_m(x)$ is also a solution of legendre Eq^n.

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)P_m(x) = 0 \quad \text{——} ③$$

multiplying Eq^n ② by $P_m(x)$ and Eq^n ③ by $P_n(x)$, we get.

$$② \Rightarrow \quad P_m(x)\cdot\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_m(x)\cdot P_n(x) = 0 \quad \text{——} ④$$

$$③ \Rightarrow \quad P_n(x)\cdot\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)P_m(x)\cdot P_n(x) = 0 \quad \text{——} ⑤$$

Eq^n ④ $-$ Eq^n ⑤

$$\Rightarrow P_m(x)\cdot\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_m(x)\cdot P_n(x) - P_n(x)\cdot\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}$$
$$- m(m+1)P_m(x)\cdot P_n(x) = 0.$$

Integrating this w.r.t 'x' from -1 to 1, we get.

$$\int_{-1}^{1} P_m(x)\left[\frac{d}{dx}(1-x^2)\frac{dP_n}{dx}\right]dx - \int_{-1}^{1} P_n(x)\left[\frac{d}{dx}(1-x^2)\frac{dP_m}{dx}\right]dx + \left[n(n+1)-m(m+1)\right]\int_{-1}^{1} P_m(x)\cdot P_n(x)\,dx = 0.$$

$$\Rightarrow \left[P_m(x)\cdot(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{1} - \int_{-1}^{1}\left[\frac{dP_m}{dx}\left((1-x^2)\frac{dP_n}{dx}\right)\right]dx - \left[P_n(x)\cdot(1-x^2)\frac{dP_m}{dx}\right]_{-1}^{1}$$

$$+ \int_{-1}^{1}\left[\frac{dP_n}{dx}\left((1-x^2)\frac{dP_m}{dx}\right)\right]dx + \left[n(n+1)-m(m+1)\right]\int_{-1}^{1} P_m(x)\cdot P_n(x)\,dx = 0. \quad \left[\begin{array}{l}\text{Integrating}\\ \text{by Parts}\end{array}\right]$$

$$\Rightarrow \left[n(n+1)-m(m+1)\right]\int_{-1}^{1} P_m(x)\cdot P_n(x)\,dx = 0.$$

$$\Rightarrow (n-m)(n+m+1)\int_{-1}^{1} P_m(x)\cdot P_n(x)\,dx = 0$$

Since m and n are non-negative integers, $m+n+1 \neq 0$.

So if $n-m \neq 0$, i.e., $m \neq n$. we get $\int_{-1}^{1} P_m(x) \cdot P_n(x) dx = 0$.

This is known as the orthogonality property of legendre polynomials.

Case (ii): when $m=n$

we know that the generating function for $P_n(x)$, we have.

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{——— (1)}$$

$$(1-2xt+t^2)^{-1/2} = \sum_{m=0}^{\infty} P_m(x) t^m \quad \text{——— (2)}$$

Multiplying Eq^{n}s (1) & (2), we get

$$(1)\times(2) \Rightarrow (1-2xt+t^2)^{-\frac{1}{2}+(-\frac{1}{2})} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_n(x) \cdot P_m(x) t^{m+n}.$$

$$\Rightarrow \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_n(x) \cdot P_m(x) t^{m+n} = (1-2xt+t^2)^{-1} = \frac{1}{(1-2xt+t^2)}$$

Now integrating w.r.t 'x', between the limits -1 & 1.

$$\Rightarrow \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-1}^{1} P_n(x) \cdot P_m(x) t^{m+n} dx = \int_{-1}^{1} \frac{1}{(1-2xt+t^2)} dx.$$

If $m=n$.

$$\Rightarrow \sum_{n=0}^{\infty} \int_{-1}^{1} P_n(x) \cdot P_n(x) t^{2n} dx = \left(\frac{-1}{2t}\right) \left[\log(1-2xt+t^2)\right]_{-1}^{1}$$

$$\Rightarrow \sum_{n=0}^{\infty} \int_{-1}^{1} (P_n(x))^2 t^{2n} dx = \left(\frac{-1}{2t}\right)\left[\log(1-2t+t^2) - \log(1+2t+t^2)\right]$$

$$= \left(\frac{-1}{2t}\right)\left[\log(1-t)^2 - \log(1+t)^2\right]$$

$$= \left(-\frac{1}{2t}\right)\cdot 2\left[\log(1-t) - \log(1+t)\right]$$

$$= \left(\frac{-1}{t}\right)\left[-(\log(1+t) - \log(1-t))\right]$$

$$= \frac{1}{t}\left[\log(1+t) - \log(1-t)\right]$$

$$= \frac{1}{t}\left[\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots\right) - \left(-t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots\right)\right]$$

$$= \frac{1}{t}\left[(t+t) + \left(\frac{t^2}{2} - \frac{t^2}{2}\right) + \left(\frac{t^3}{3} + \frac{t^3}{3}\right) + \cdots\right]$$

$$= \frac{1}{t}\left[2\left(t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots\right)\right]$$

$$= \frac{2}{t}\left[t\left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots + \frac{t^{2n}}{(2n+1)}\right)\right]$$

$$= 2\cdot\sum_{n=0}^{\infty} \frac{1}{2n+1} t^{2n}$$

$$\therefore \sum_{n=0}^{\infty} \int_{-1}^{1} [P_n(x)]^2 dx \cdot t^{2n} = 2\cdot\sum_{n=0}^{\infty} \frac{1}{2n+1} t^{2n}.$$

Now equating the coefficient of t^{2n} on B.S. we get

$$\int_{-1}^{1} (P_n(x))^2 dx = \frac{2}{2n+1}$$

i.e., if $m=n$, $\displaystyle\int_{-1}^{1} P_m(x)\cdot P_n(x) dx = \frac{2}{2n+1}$.

//

problems:-

1) **Beltrami's result:**

To prove that $(2n+1)(x^2-1)P_n'(x) = n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$

(or) $(2n+1)(1-x^2)P_n'(x) = n(n+1)\left[P_{n-1}(x) - P_{n+1}(x)\right]$

proof :- we have the recurrence relations

$$(1-x^2)P_n'(x) = n\left[P_{n-1}(x) - xP_n(x)\right] \text{ (Relation V)} \quad \text{—— } ①$$

and $$(1-x^2)P_n'(x) = (n+1)\left[xP_n(x) - P_{n+1}(x)\right] \text{ (Relation VI)} \quad \text{—— } ②$$

from ①, $\Rightarrow \frac{1}{n}(1-x^2)P_n'(x) = P_{n-1}(x) - xP_n(x)$

$$\Rightarrow xP_n(x) = P_{n-1}(x) - \frac{1}{n}(1-x^2)P_n'(x). \quad \text{—— } ③$$

sub. eqⁿ ③ in eqⁿ ②, we get

② $\Rightarrow (1-x^2)P_n'(x) = (n+1)\left[P_{n-1}(x) - \frac{1}{n}(1-x^2)P_n'(x) - P_{n+1}(x)\right]$

$\Rightarrow (1-x^2)P_n'(x) = (n+1)\left[P_{n-1}(x) - P_{n+1}(x)\right] - \frac{(n+1)}{n}(1-x^2)P_n'(x)$

$\Rightarrow (1-x^2)P_n'(x) + \frac{(n+1)}{n}(1-x^2)P_n'(x) = (n+1)\left[P_{n-1}(x) - P_{n+1}(x)\right]$

$\Rightarrow (1-x^2)\left[1 + \frac{n+1}{n}\right]P_n'(x) = (n+1)\left[-\left(P_{n+1}(x) - P_{n-1}(x)\right)\right]$

$\Rightarrow (1-x^2)\frac{(2n+1)}{n}P_n'(x) = -(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$

$\Rightarrow (2n+1)(x^2-1)P_n'(x) = n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$

this is Beltrami's result.

(2) prove that $P_{n+1}' + P_n' = P_0 + 3P_1 + 5P_2 + \cdots + (2n+1)P_n$

Sol: we have the recurrence relation

$$(2n+1)P_n = P_{n+1}' - P_{n-1}' \longrightarrow (1)$$

Putting $n = 1, 2, 3, \ldots$ in (1), we get

$n=1$, $3P_1 = P_2' - P_0'$

$n=2$, $5P_2 = P_3' - P_1'$

$n=3$, $7P_3 = P_4' - P_2'$

and so on ...

$(2n-3)P_{n-2} = P_{n-1}' - P_{n-3}'$

$(2n-1)P_{n-1} = P_n' - P_{n-2}'$

$(2n+1)P_n = P_{n+1}' - P_{n-1}'$

Adding all the above relations, we get

$$3P_1 + 5P_2 + 7P_3 + \cdots + (2n-1)P_{n-1} + (2n+1)P_n = \cancel{P_2'} - P_0' + \cancel{P_3'} - P_1' + \cancel{P_4'} - \cancel{P_2'} + P_n' - \cancel{P_{n-2}'} + P_{n+1}' - \cancel{P_{n-1}'}.$$

$$\Rightarrow 3P_1 + 5P_2 + \cdots + (2n+1)P_n = -P_0' - P_1' + P_{n+1}' + P_n'$$

$$\Rightarrow 3P_1 + 5P_2 + \cdots + (2n+1)P_n = 0 - P_0 + P_{n+1}' + P_n' \quad [\because P_0' = 0 \ \& \ P_1' = P_0]$$

$$\Rightarrow P_0 + 3P_1 + 5P_2 + \cdots + (2n+1)P_n = P_{n+1}' + P_n'$$

$$\therefore P_0 + 3P_1 + 5P_2 + \cdots + (2n+1)P_n = P_{n+1}' + P_n'.$$

3) Prove that $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$

Sol^n: We have the recurrence relations.

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x) \quad \text{(Relation II)} \quad \text{——} (1)$$

and $$(2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x) \quad \text{(Relation III)} \quad \text{——} (2)$$

Eq^n (2) − Eq^n (1) gives

$$(2n+1)P_n(x) - nP_n(x) = P_{n+1}'(x) - \cancel{P_{n-1}'(x)} - xP_n'(x) + \cancel{P_{n-1}'(x)}$$

$$(2n+1-n)P_n(x) = P_{n+1}'(x) - xP_n'(x)$$

$$(n+1)P_n(x) = P_{n+1}'(x) - xP_n'(x)$$

replacing 'n' by $(n-1)$.

$$(n-\cancel{1}+\cancel{1})P_{n-1}(x) = P_{n-\cancel{1}+\cancel{1}}'(x) - xP_{n-1}'(x)$$

$$nP_{n-1}(x) = P_n'(x) - xP_{n-1}'(x)$$

$$\boxed{P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)}$$

4) Prove that $\int_{-1}^{1} xP_n(x) \cdot P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$

We have the recurrence relation

$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$ —① [Relation – I]

Multiplying eqⁿ ① bothsides with $P_{n-1}(x)$

①⇒ $(2n+1)xP_n(x)P_{n-1}(x) = (n+1)P_{n+1}(x)P_{n-1}(x) + nP_{n-1}(x)\cdot P_{n-1}(x)$

Now Integrating this w.r.t 'x' from -1 to 1, we get

$$\Rightarrow \int_{-1}^{1}(2n+1)xP_n(x)P_{n-1}(x)dx = \int_{-1}^{1}(n+1)P_{n+1}(x)P_{n-1}(x)dx + \int_{-1}^{1} nP_{n-1}^2(x)dx$$

By using orthogonality of legendre polynomial.

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & \text{if } m\neq n \\ \frac{2}{2n+1}, & \text{if } m=n \end{cases}$$

$$\Rightarrow (2n+1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = (n+1)(0) + n\left(\frac{2}{2(n-1)+1}\right)$$

$$\Rightarrow (2n+1)\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{2n-1}$$

$$\Rightarrow \int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{(2n+1)(2n-1)} = \frac{2n}{4n^2-1}$$

$$\boxed{\therefore \int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}}$$

5) Prove that $\displaystyle\int_{-1}^{1}(x^2-1)P_{n+1}(x)P_n'(x)dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$

solⁿ: we have the Beltrami's result

$$(2n+1)(x^2-1)P_n'(x) = n(n+1)\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

$$\Rightarrow (x^2-1)P_n'(x) = \frac{n(n+1)}{(2n+1)}\left[P_{n+1}(x) - P_{n-1}(x)\right] \quad \text{— ① say}$$

Multiplying eq^n ① on b.s with $P_{n+1}(x)$

$$(x^2-1)P_{n+1}(x)P_n'(x) = \frac{n(n+1)}{(2n+1)}\left[P_{n+1}(x)P_{n+1}(x) - P_{n+1}(x)P_{n-1}(x)\right]$$

$$\Rightarrow \int_{-1}^{1}(x^2-1)P_{n+1}(x)P_n'(x)\,dx = \frac{n(n+1)}{(2n+1)}\int_{-1}^{1}P_{n+1}(x)P_{n+1}(x)\,dx - \frac{n(n+1)}{(2n+1)}\int_{-1}^{1}P_{n+1}(x)\cdot P_{n-1}(x)\,dx$$

By using orthogonality of legendre polynomial

$$\int_{-1}^{1}P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m\neq n \\ \frac{2}{2n+1}, & \text{if } m=n \end{cases}$$

$$\Rightarrow \int_{-1}^{1}(x^2-1)P_{n+1}(x)P_n'(x)\,dx = \frac{n(n+1)}{2n+1}\cdot\frac{2}{2(n+1)+1} - \frac{n(n+1)}{2n+1}(0)$$

$$\Rightarrow \int_{-1}^{1}(x^2-1)P_{n+1}(x)P_n'(x)\,dx = \frac{n(n+1)}{(2n+1)}\cdot\frac{2}{2n+3}$$

$$\therefore \boxed{\int_{-1}^{1}(x^2-1)P_{n+1}(x)P_n'(x)\,dx = \frac{2n(n+1)}{(2n+1)(2n+3)}}$$

1) Prove that $\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx = \begin{cases} 0, & \text{if } m\neq n \\ \frac{2n(n+1)}{2n+1}, & \text{if } m=n \end{cases}$

Solⁿ: case(i): $m\neq n$:-

We know that the legendre differential eqⁿ is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{—— ① say}$$

Now we know that $P_m(x)$ is a solution of eqⁿ ①

$$① \Rightarrow (1-x^2)y'' - 2xy' + m(m+1)y = 0$$

$$\Rightarrow \frac{d}{dx}\left\{(1-x^2)y'\right\} + m(m+1)y = 0$$

$$\Rightarrow \frac{d}{dx}\left\{(1-x^2)P_m'(x)\right\} + m(m+1)P_m(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left\{(1-x^2)P_m'(x)\right\} = -m(m+1)P_m(x) \quad \text{—— ② say}$$

Now $\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx = \int_{-1}^{1}\left\{(1-x^2)P_m'(x)\right\}\cdot P_n'(x)\,dx$

$$= \left[(1-x^2)P_m'(x)\cdot P_n(x)\right]_{-1}^{1} - \int_{-1}^{1}\left[\frac{d}{dx}\left\{(1-x^2)P_m'(x)\right\}\cdot P_n(x)\right]dx$$

[∵ using integrating by parts $\int uv = u\int v - \int u'\int v$]

$$= 0 - \int_{-1}^{1}\left(-m(m+1)P_m(x)\right)\cdot P_n(x)\,dx$$

[∵ using ②]

$$= m(m+1)\int_{-1}^{1}P_m(x)\cdot P_n(x)\,dx$$

$$= m(m+1)[0]$$

[∵ By using the orthogonality of legendre's polynomial $\int_{-1}^{1}P_m(x)P_n(x)dx = \begin{cases} 0, & \text{if } m\neq n \\ \frac{2}{2n+1}, & \text{if } m=n \end{cases}$]

$$\therefore \int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx = 0, \quad \text{for } m\neq n$$

case(ii): $m=n$:-

If $m=n$ then

$$\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx = \int_{-1}^{1}(1-x^2)P_n'(x)P_n'(x)dx$$

$$= \int_{-1}^{1}\{(1-x^2)P_n'(x)\}\cdot P_n'(x)\,dx$$

$$= \Big[(1-x^2)P_n'(x)P_n(x)\Big]_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}\{(1-x^2)P_n'(x)\}\,P_n(x)\,dx \quad [\because \text{By integrating by parts}]$$

$$= 0 - \int_{-1}^{1}\{(1-x^2)P_n''(x) - 2xP_n'(x)\}\,P_n(x)dx \quad [\because \tfrac{d}{dx}(uv) = uv'+vu']$$

$$= -\int_{-1}^{1}\Big(\big((1-x^2)P_n''(x) - 2xP_n'(x)\big)P_n(x)dx$$

$$\therefore \int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx = -\int_{-1}^{1}\Big((1-x^2)P_n''(x) - 2xP_n'(x)\Big)P_n(x)dx \quad \text{—③}$$

From eq^n ①, $(1-x^2)y''-2xy'+n(n+1)y=0$

$P_n(x)$ is a sol^n of this eq^n, so

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

$$(1-x^2)P_n''(x) - 2xP_n'(x) = -n(n+1)P_n(x) \quad \text{—④}$$

Sub eq^n ④ in eq^n ③, we get

$$③ \Rightarrow \int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx = -\int_{-1}^{1}\big(-n(n+1)P_n(x)\big)P_n(x)dx$$

$$= n(n+1) \int_{-1}^{1} P_n^2(x)\, dx$$

$$= n(n+1) \cdot \frac{2}{(2n+1)}$$

[$\because$ Using orthogonality of Legendre polynomials

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

$$= \frac{2n(n+1)}{(2n+1)}$$

$$\int_{-1}^{1} (1-x^2) P_m'(x) P_n'(x)\, dx = \frac{2n(n+1)}{(2n+1)}, \text{ for } m = n$$

Hence $$\int_{-1}^{1} (1-x^2) P_m'(x) P_n'(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2n(n+1)}{(2n+1)}, & \text{if } m = n \end{cases}$$

8) Obtain (or) Find the first three terms of the following function in terms of Legendre polynomial

$$f(x) = \begin{cases} 0, & \text{if } -1 \leq x < 0 \\ x, & \text{if } 0 \leq x \leq 1. \end{cases}$$

Sol^n: We know that if $f(x)$ is the polynomial of degree n

then $f(x) = \sum_{r=0}^{n} C_r P_r(x), \quad -1 < x < 1$

where $C_r = \left(r+\frac{1}{2}\right) \int_{-1}^{1} f(x) P_r(x)\, dx$

Given $$f(x) = \begin{cases} 0, & \text{if } -1 \leq x < 0 \\ x, & \text{if } 0 \leq x \leq 1 \end{cases}$$

Now

$$C_r = \left(r+\frac{1}{2}\right) \int_{-1}^{1} f(x) P_r(x)\, dx$$

$$= (r+\tfrac{1}{2})\left[\int_{-1}^{0} f(x)P_r(x)dx + \int_{0}^{1} f(x)P_r(x)dx\right]$$

$$= (r+\tfrac{1}{2})\left[\int_{-1}^{0} 0\cdot P_r(x)dx + \int_{0}^{1} x\cdot P_r(x)dx\right]$$

$$\boxed{C_r = (r+\tfrac{1}{2})\int_{0}^{1} x\cdot P_r(x)dx}$$

$$f(x) = \sum_{r=0}^{n} C_r P_r(x) = C_0P_0(x) + C_1P_1(x) + C_2P_2(x) + \cdots \quad \text{——} (2).$$

Now putting $r=0,1,2,3,\ldots$ in eq^n (1), we get

$$r=0, \quad C_0 = (0+\tfrac{1}{2})\int_{0}^{1} x\cdot P_0(x)dx = \frac{1}{2}\int_{0}^{1} x(1)dx \quad [\because P_0(x) = 1]$$

$$= \frac{1}{2}\left[\int_{0}^{1} x\,dx\right] = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 = \left[\frac{x^2}{4}\right]_0^1$$

$$\boxed{C_0 = \frac{1}{4}}$$

$$r=1, \quad C_1 = (1+\tfrac{1}{2})\int_{0}^{1} x\cdot P_1(x)dx = \frac{3}{2}\int_{0}^{1} x(x)dx \quad [\because P_1(x) = x]$$

$$= \frac{3}{2}\int_{0}^{1} x^2dx = \frac{\cancel{3}}{2}\left(\frac{x^3}{\cancel{3}}\right)_0^1 = \frac{1}{2}(1-0) = \frac{1}{2}$$

$$\boxed{C_1 = \frac{1}{2}}$$

$$r=2, \quad C_2 = (2+\tfrac{1}{2})\int_{0}^{1} x P_2(x)dx$$

$$= \frac{5}{2}\int_{0}^{1} x\left(\frac{3x^2-1}{2}\right)dx \quad \left[\because P_2(x) = \frac{3x^2-1}{2}\right]$$

$$= \frac{5}{4}\int_{0}^{1}(3x^3-x)dx$$

$$= \frac{5}{4}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_0^1$$

$$= \frac{5}{4}\left[\frac{3}{4}-\frac{1}{2}\right]$$

$$\therefore \boxed{c_2 = \frac{5}{16}}$$

Now, sub the values of $c_0, c_1, c_2, \dots$ in eqⁿ ②, we get

$$f(x) = \left(\frac{1}{4}\right)P_0(x) + \left(\frac{1}{2}\right)P_1(x) + \frac{5}{16}\left(P_2(x)\right) + \dots$$

$$\therefore f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) + \dots$$

Expansion of x^n in Legendre's polynomials :-

Let $x^n = a_n P_n(x) + a_{n-2}P_{n-2}(x) + \dots + a_r P_r(x) + \dots$ —— ①

To determine the coefficients $a_n, a_{n-2}, a_{n-4}, \dots\dots$, multiply both sides by $P_m(x)$ and integrate from -1 to 1, where $m = n, n-2, n-4, \dots$

Sub the values of $a_n, a_{n-2}, a_{n-4}, \dots$ in eqⁿ ①, we get

$$x^n = \frac{n!}{3.5\dots(2n+1)}\left[(2n+1)P_n(x) + (2n-3)\frac{(2n+1)}{2}P_{n-2}(x) + (2n-7)\frac{(2n+1)(2n-1)}{2.4}P_{n-4}(x) + \dots + \frac{1}{(n+1)}P_0(x) \text{ (or) } \frac{3}{(n+2)}P_1(x)\right]$$

According as n is even or odd

1) Show that $\forall x$,

i) $x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)$

ii) $x^5 = \frac{8}{63} P_5(x) + \frac{4}{9} P_3(x) + \frac{3}{7} P_1(x)$

iii) $x^7 = \frac{16}{429} P_7(x) + \frac{8}{39} P_5(x) + \frac{14}{33} P_3(x) + \frac{1}{3} P_1(x)$

Soln: We know that

$$x^n = \frac{n!}{3\cdot 5 \cdots (2n+1)}\Big[(2n+1)P_n(x) + (2n-3)\frac{(2n+1)}{2} P_{n-2}(x) + (2n-7)\frac{(2n+1)(2n-1)}{2\cdot 4} P_{n-4}(x) + \dots + \frac{1}{(n+1)} P_0(x) \text{ (or) } \frac{3}{(n+2)} P_1(x)\Big] \quad \text{--- (1)}$$

i) Putting $n=3$ in eqn (1), we get

$$x^3 = \frac{3!}{3\cdot 5\cdot 7}\Big[(2(3)+1)P_3(x) + (2(3)-3)\frac{(2(3)+1)}{2} P_1(x)\Big]$$

$$= \frac{\cancel{3}\cdot 2}{\cancel{3}\cdot 5\cdot 7}\Big[7P_3(x) + 3\times\left(\frac{7}{2}\right)P_1(x)\Big]$$

$$= \frac{2\cdot\cancel{7}}{5\cdot\cancel{7}} P_3(x) + \frac{\cancel{2}\cdot 3\cdot\cancel{7}}{5\cdot\cancel{7}\cdot\cancel{2}} P_1(x)$$

$$\boxed{x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)}$$

ii) Putting $n=5$ in eqn (1), we get

$$x^5 = \frac{5!}{3\cdot 5\cdot 7\cdot 9\cdot 11}\Big[(11)P_5(x) + 7\frac{(11)}{2} P_3(x) + (3)\frac{(11)(9)}{2\cdot 4} P_1(x)\Big]$$

$$= \frac{\cancel{5}\cdot 4\cdot\cancel{3}\cdot 2\cdot 1}{\cancel{3}\cdot\cancel{5}\cdot 7\cdot 9\cdot\cancel{11}}\times\cancel{11} P_5(x) + \frac{\cancel{5}\cdot 4\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{3}\cdot\cancel{5}\cdot\cancel{7}\cdot 9\cdot\cancel{11}}\times\frac{\cancel{7}\cdot\cancel{11}}{\cancel{2}} P_3(x) +$$

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$$\frac{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}{\cancel{3}\cdot\cancel{5}\cdot 7\cdot\cancel{9}\cdot\cancel{11}} \times \frac{3\cdot\cancel{11}\cdot\cancel{9}}{\cancel{2}\cdot\cancel{4}} P_1(x)$$

$$= \frac{8}{63} P_5(x) + \frac{4}{9} P_3(x) + \frac{3}{7} P_1(x)$$

iii) $$x^n = \frac{n!}{3\cdot 5 \cdots (2n+1)} \Big[(2n+1) P_n(x) + (2n-3)\frac{(2n+1)}{2} P_{n-2}(x) + (2n-7) \frac{(2n+1)(2n-1)}{2\cdot 4} P_{n-4}(x) + (2n-11)\frac{(2n+1)(2n-1)(2n-3)}{2\cdot 4\cdot 6} P_{n-6}(x) + \cdots \Big]$$

Putting $n=7$ in x^n:

$$x^7 = \frac{7!}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15} \Big[15 P_7(x) + (11)\frac{15}{2} P_5(x) + (7)\frac{(15)(13)}{2\cdot 4} P_3(x) + (3)\frac{(15)(13)(11)}{2\cdot 4\cdot 6} P_1(x) \Big]$$

$$x^7 = \frac{\cancel{7}\cdot 6\cdot \cancel{5}\cdot 4\cdot \cancel{3}\cdot 2\cdot 1}{\cancel{3}\cdot\cancel{5}\cdot\cancel{7}\cdot 9\cdot 11\cdot 13\cdot 15} \Big[15 P_7(x) + \frac{11\cdot 15}{2} P_5(x) + \frac{7\cdot 15\cdot 13}{2\cdot 4} P_3(x) + \frac{3\cdot 15\cdot 13\cdot 11}{2\cdot 4\cdot 6} P_1(x) \Big]$$

$$x^7 = \frac{\overset{2}{\cancel{6}}\cdot 4\cdot 2\cdot \cancel{15}}{\underset{3}{\cancel{9}}\cdot 11\cdot 13\cdot \cancel{15}} P_7(x) + \frac{\overset{2}{\cancel{6}}\cdot 4\cdot \cancel{2}\cdot\cancel{11}\cdot\cancel{15}}{\underset{3}{\cancel{9}}\cdot\cancel{11}\cdot 13\cdot\cancel{15}\cdot\cancel{2}} P_5(x) + \frac{\overset{2}{\cancel{6}}\cdot\cancel{4}\cdot\cancel{2}\cdot 7\cdot\cancel{15}\cdot\cancel{13}}{\underset{3}{\cancel{9}}\cdot 11\cdot\cancel{13}\cdot\cancel{15}\cdot\cancel{2}\cdot\cancel{4}} P_3(x) +$$

$$\frac{\cancel{6}\cdot\cancel{4}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{15}\cdot\cancel{13}\cdot\cancel{11}}{\underset{3}{\cancel{9}}\cdot\cancel{11}\cdot\cancel{13}\cdot\cancel{15}\cdot\cancel{2}\cdot\cancel{4}\cdot\cancel{6}} P_1(x)$$

$$\boxed{x^7 = \frac{16}{429} P_7(x) + \frac{8}{39} P_5(x) + \frac{14}{33} P_3(x) + \frac{1}{3} P_1(x)}$$

Module - II

Bessel's Polynomials

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

1. General solution
2. Bessel functions
3. Recurrence relations
4. Generating function
5. orthogonality.
6. Related problems.

Legendre's polynomials.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

1. General solution
2. Legendre's polynomials.
3. Recurrence relations.
4. Generating function
5. orthogonality.
6. Related problems.

Module - II

Legendre's & Bessel's Polynomials

Bessel's equation :-

The differential equation

$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$ where n is a positive constant, is known as Bessel's equation. Its particular solutions are called Bessel functions of order n.

General solution of Bessel's differential equation :-

The bessel's differential eqn is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \longrightarrow ①.$$

Compare eqn ① with $P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0$.

where $P_0(x) = x^2 \Rightarrow$ at $x=0$, $P_0(x)=0$.

$\Rightarrow x=0$ is singular point.

Divide eqn ① with x^2,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\frac{x^2-n^2}{x^2}\right)y = 0$$

which is of the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$.

where $P(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2-n^2}{x^2}$.

$x\,P(x) = 1$ & $x^2\,Q(x) = x^2-n^2$ are analytic at $x=0$.

$\therefore x=0$ is regular singular point.

To obtain its solution,

let $\quad y = \sum_{k=0}^{\infty} a_k x^{k+m}$, $(a_0 \neq 0)$.

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k (k+m) x^{k+m-1}$$

and $\dfrac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k (k+m)(k+m-1) x^{k+m-2}$.

substituting these values in ①, we have

$$x^2 \sum_{k=0}^{\infty} a_k (k+m)(k+m-1) x^{k+m-2} + x \sum_{k=0}^{\infty} a_k (k+m) x^{k+m-1}$$

$$+ (x^2 - n^2) \sum_{k=0}^{\infty} a_k x^{k+m} = 0.$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \Big[\big\{ (k+m)(k+m-1) + (k+m) - n^2 \big\} x^{k+m} + x^{k+m+2} \Big] = 0.$$

$(k^2+km-k+mk+m^2-m)$

$(k+m)^2-(k+m)$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \Big[\big\{ (k+m)^2 - \cancel{(k+m)} + \cancel{(k+m)} - n^2 \big\} x^{k+m} + x^{k+m+2} \Big] = 0.$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \big[(k+m)^2 - n^2 \big] x^{k+m} + \sum_{k=0}^{\infty} a_k x^{k+m+2} = 0. \longrightarrow ②$$

To get indicial equation, equate the coefficient of lowest degree of x to zero.

i.e, the coefficient of x^m to zero [$\because$ when $k=0$ in ②]

$$a_0 (m^2 - n^2) = 0. \quad [\because a_0 \neq 0]$$

$$m^2 - n^2 = 0 \Rightarrow \text{which is indicial equation.}$$

$$\Rightarrow m = \pm n.$$

The roots are real and distinct.

Also equating the coefficient of x^{m+1} to zero, we get (i.e, when $k=1$). ②

$$a_1 \left[(1+m)^2 - n^2\right] = 0$$

$$\Rightarrow a_1 \left[\{(1+m)-n\}\{(1+m)+n\}\right] = 0.$$

$$\Rightarrow a_1 = 0 \quad [\because m = \pm n].$$

Now equating the coefficients of general term i.e, x^{m+k} in ② to zero, we have

$$② \Rightarrow \sum_{k=0}^{\infty} a_k \left[(k+m)^2 - n^2\right] x^{k+m} + \sum_{k=2}^{\infty} a_{k-2} x^{k+m} = 0$$

$$a_k \{(k+m)^2 - n^2\} + a_{k-2} = 0, \quad k \geq 2$$

$$a_k = -\frac{a_{k-2}}{(k+m-n)(k+m+n)} \longrightarrow ③$$

putting $k=3$, $a_3 = -\dfrac{a_1}{(3+m-n)(3+m+n)} = 0 \quad (\because a_1 = 0).$

|||^{ly} putting $k = 5, 7, 9, \ldots$ we can find that

$$a_5 = a_7 = a_9 = \cdots = 0$$

Hence, we ~~are~~ have to evaluate only $a_2, a_4, a_6, \ldots$

Now we have two cases

Case ① :– when $m = n$.

In this case from ③, we get

$$a_k = -\frac{a_{k-2}}{(2n+k)k}$$

when $k=2$, $a_2 = -\dfrac{a_0}{(2n+2)2} = -\dfrac{a_0}{2(2n+2)}$

when $k=4$, $a_4 = -\dfrac{a_2}{(4+2n)4} = \dfrac{a_0}{2\cdot 4(2n+2)(2n+4)}$

$$\Rightarrow a_{2r} = \frac{(-1)^r a_0}{2\cdot 4\cdot 6\cdots(2r)\,(2n+2)(2n+4)\cdots(2n+2r)} = \frac{(-1)^r a_0}{2^r(1\cdot 2\cdot 3\cdots r)\,2^r(n+1)(n+2)\cdots(n+r)}.$$

when $k = 2r$,

we get $a_{2r} = \dfrac{(-1)^r a_0}{2^{2r} \cdot r! \, (n+1)(n+2) \cdots \cdots (n+r)}$.

substituting all these values in

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}.$$

$$y = x^m \left[a_0 + a_1 x + a_2 x^2 + \cdots \right]$$

$$= x^n \left[a_0 + a_1 x + a_2 x^2 + \cdots \right] \quad [\because m = n]..$$

$$y = x^n \left[a_0 + 0 \cdot x + \left(-\frac{a_0}{2(2n+2)} \right) x^2 + \cdots \cdots + \frac{(-1)^r a_0 \quad x^{2r}}{2^{2r} \cdot r! \, (n+1)(n+2) \cdots (n+r)} \right]$$

$$y = a_0 x^n \left[1 - \frac{1}{2(2n+2)} \cdot x^2 + \cdots \cdots + \frac{(-1)^r \quad x^{2r}}{2^{2r} \cdot r! \, (n+1)(n+2) \cdots (n+r)} + \cdots \right]$$

$$y = a_0 \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{2r} \cdot r! \, (n+1)(n+2) \cdots (n+r)}.$$

$$y = a_0 \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{2r} \cdot r! \, (n+1)(n+2) \cdots (n+r)} \cdot \frac{2^n \gamma(n+1)}{2^n \gamma(n+1)}.$$

$$y = a_0 2^n \gamma(n+1) \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \gamma(n+1) \cdot (n+1)(n+2) \cdots (n+r)} \cdot \left(\frac{x}{2} \right)^{n+2r}.$$

$$\left[\begin{array}{l} \because \gamma(n+1) = n! \\ \gamma(n+r+1) = (n+r)! \end{array} \right.$$

$$y = a_0 2^n \gamma(n+1) \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \, \gamma(n+r+1)} \left(\frac{x}{2} \right)^{n+2r}.$$

If $a_0 = \dfrac{1}{2^n \gamma(n+1)}$, then this particular solution is known as Bessel's function of first kind of order n. It is denoted by $J_n(x)$.

Thus,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\,\Gamma(n+r+1)} \longrightarrow (4)$$

Case (2) :- when $m = -n$,

Replacing n by $-n$ in (4), the corresponding solution of (1) in series is

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r-n}}{r!\,\Gamma(-n+r+1)} \longrightarrow (5)$$

which is known as Bessel function of the first kind of order $-n$.

Thus $J_n(x)$ and $J_{-n}(x)$ are two independent solutions of (1), where n is not an integer.

$\therefore$ The general solution of (1) is

$$y = c_1 J_n(x) + c_2 J_{-n}(x).$$ where c_1 & c_2 are arbitrary constants.

Bessel functions

Bessel functions of orders 0 and 1 :-

we have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{n+2r}}{r!\, \gamma(n+r+1)}$.

① Bessel function of order "0" is given by

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{r!\, \gamma(r+1)}$$

$$= \frac{1}{\gamma(1)} + \frac{(-1)\cdot\left(\frac{x}{2}\right)^2}{\gamma(2)} + \frac{\left(\frac{x}{2}\right)^4}{2!\,\gamma(3)} + \cdots$$

$[\gamma(n+1) = n\gamma(n)]$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 \cdot 2 \cdot 2} + \cdots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \cdots$$

$\gamma(2) = \gamma(1+1)$
$= 1\gamma(1) = 1$

$\gamma(3) = \gamma(2+1)$
$= 2\gamma(2)$
$= 2\cdot\gamma(1+1)$
$= 2.$

② Bessel function of order "1" is

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\, \gamma(2+r)}$$

$$= \frac{1}{\gamma(2)}\cdot\left(\frac{x}{2}\right) - \frac{1}{\gamma(3)}\cdot\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,\gamma(4)}\left(\frac{x}{2}\right)^5 + \cdots$$

$$= \frac{x}{2} - \frac{x^3}{2\cdot 2^3} + \frac{x^5}{2!\,(3\cdot 2\cdot 1)\, 2^5} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} + \cdots$$

Note :- ① $\gamma(n+1) = n!$ ② $\gamma(n) = (n-1)\gamma(n-1)$

③ $\gamma(n+1) = n\gamma(n)$.

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Recurrence formulae for $J_n(x)$:-

① $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \longrightarrow$ Recurrence formula ①.

Sol :-

we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r} \longrightarrow ①$$

multiplying ① by x^n, we get

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}.$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, x^n \cdot x^{n+2r}}{r!\,\Gamma(n+r+1)\, 2^{n+2r}}$$

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{2(n+r)}}{r!\,\Gamma(n+r+1)\, 2^{n+2r}},$$ differentiating both sides

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r)\, x^{2(n+r)-1}}{r!\,\Gamma(n+r+1)\, 2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2\cancel{(n+r)}\, x^{2n+2r-1}}{r!\,\cancel{(n+r)}\,\Gamma(n+r)\cdot 2^{n+2r}} \quad \left[\because \Gamma(n+1) = n\Gamma(n)\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, x^n \cdot x^{n+2r-1}}{r!\,\Gamma(n+r)\, 2^{n+2r} \cdot 2^{-1}}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma((n-1)+r+1)} \cdot \left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$= x^n J_{n-1}(x) \quad [\because ①].$$

—×—

(2) $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

$\longrightarrow$ Recurrence formula (2).

Sol:— we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}.$$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot x^{-n} \cdot x^{n+2r}}{r!\,\Gamma(n+r+1)\, 2^{n+2r}}$$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!\,\Gamma(n+r+1)\, 2^{n+2r}},$$

differentiating both sides

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot (2r) \cdot x^{2r-1}}{r!\,\Gamma(n+r+1)\, 2^{n+2r}}.$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r\, 2\cancel{r} \cdot x^{2r-1}}{\cancel{r}(r-1)!\,\Gamma(n+r+1)\, 2^{n+2r}}$$ [$\because$ for $r=0$, the first term in R.H.S is zero]

$$= \sum_{r-1=0}^{\infty} \frac{(-1)^r\, x^{2r-1}}{(r-1)!\,\Gamma(n+r+1)\, 2^{n+2r} \cdot 2^{-1}}$$

put $r-1=s$ & $r=s+1$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{s+1}\, x^{2(s+1)-1}\, x^{-n} \cdot x^{n}}{s!\,\Gamma(n+(s+1)+1)} \cdot \frac{1}{2^{n+2r-1}}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)(-1)^s\, x^{2(s+1)-1}\, x^{-n} \cdot x^{n}}{s!\,\Gamma((n+1)+s+1)} \cdot \frac{1}{2^{n+2(s+1)-1}}$$

$$= -x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma((n+1)+s+1)} \cdot \left(\frac{x}{2}\right)^{n+2(s+1)-1}.$$

$$= -x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma((n+1)+s+1)} \cdot \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= -x^{-n} J_{n+1}(x)$$

———x———

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Recurrence relation (3) :-

(3) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(or)

$$\frac{n}{x} J_n(x) - J_n'(x) = J_{n+1}(x).$$

proof :-

we know that $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$

differentiating both sides w.r.to x,

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) . x^{n+2r-1}}{r!\,\gamma(n+r+1)\, 2^{n+2r}}$$

multiplying both sides ~~w.~~ with x,

$$x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r) . x^{n+2r-1} (x^{-1} x)}{r!\,\gamma(n+r+1)\, 2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} n . \frac{(-1)^r}{r!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r . 2r}{r!\,\gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= n . \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r . 2\,r}{r(r-1)!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r . 2}{(r-1)!\,\gamma(n+r+1)} \cdot \frac{x^{n+2r}}{2^{n+2r}}$$

$$= n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r (x . x^{-1})}{(r-1)!\,\gamma(n+r+1)} \frac{x^{n+2r}}{2^{n+2r-1}}$$

$$= n J_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)!\,\gamma(n+r+1)} \cdot \frac{x^{n+2r-1}}{2^{n+2r-1}}$$

$$= n J_n(x) + x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

put $r-1 = s$ & $r = s+1$

$$= n J_n(x) + x \sum_{r=1}^{\infty} \frac{(-1)^{s+1}}{s!\,\gamma(n+(s+1)+1)} \cdot \left(\frac{x}{2}\right)^{n+2(s+1)-1}$$

$$= n J_n(x) + x \sum_{r-1=0}^{\infty} \frac{(-1)(-1)^s}{s!\,\gamma((n+1)+s+1)} \cdot \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= n J_n(x) - x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\gamma[(n+1)+s+1]} \cdot \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= n J_n(x) - x J_{n+1}(x).$$

$$\therefore \quad x J_n'(x) = n J_n(x) - x J_{n+1}(x)$$

—x—

Recurrence relation ④ :-

④ $$x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

(or)

$$\frac{n}{x} J_n(x) + J_n'(x) = J_{n-1}(x).$$

Proof :- we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}.$$

differentiating both sides w.r.to x,

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r) \cdot x^{(n+2r)-1}}{r!\,\gamma(n+r+1)\;2^{n+2r}}$$

multiplying both sides w.r.to x,

$$x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)\, x^{n+2r-1}}{r!\, \gamma(n+\cancel{2}r+1)\; 2^{n+2r}} \cdot x$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\, \gamma(n+\cancel{2}r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+n-n+2r)}{r!\, \gamma(n+\cancel{2}r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [(-n)+(2n+2r)]}{r!\, \gamma(n+\cancel{2}r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= (-n)\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r\, 2\cancel{(n+r)}}{r!\,\cancel{(n+r)}\,\gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r}$$

$$[\because \gamma(n+1) = n\gamma(n)]$$

$$x J_n'(x) = -n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r\, 2}{r!\, \gamma(n+r)} \cdot \frac{x^{n+2r}}{2^{n+2r}}$$

$$= -n J_n(x) + \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(x \cdot x^{-1})}{r!\, \gamma(n+r)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n J_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \gamma(n+r)} \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -n J_n(x) + x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \gamma((n-1)+r+1)} \cdot \left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

—— x ——

Recurrence relation ⑤ :-

⑤ $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

proof:-

we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

differentiating w.r.to x,

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+r+1)} \cdot \frac{(n+2r) \cdot x^{(n+2r)-1}}{2^{n+2r}}$$

multiplying both sides with 2,

$$2J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+2r)}{r!\,\gamma(n+r+1)} \cdot \frac{x^{n+2r-1}}{2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [(n+r)+r]}{r!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\,\gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r r}{r!\,\gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cancel{(n+r)}}{r!\,\cancel{(n+r)}\,\gamma(n+r)} \cdot \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \cancel{r}}{(r-1)!\,\cancel{r}\,\gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma((n-1)+r+1)} \left(\frac{x}{2}\right)^{(n-1)+2r} + \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\gamma(n+(s+1)+1)} \left(\frac{x}{2}\right)^{n+2(s+1)-1}$$

[put $r-1=s$ & $r=s+1$]

$$= J_{n-1}(x) + \sum_{s=0}^{\infty} \frac{(-1)(-1)^s}{s!\,\gamma((n+1)+s+1)} \cdot \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= J_{n-1}(x) - J_{n+1}(x)$$

$$\therefore 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

— x —

Recurrence relation (6) :-

(6) $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x) \right].$

Sol :- $J_n(x) = \frac{x}{2n}\left\{ J_{n-1}(x) + J_{n+1}(x) \right\}.$ (6a)

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r},$$

$$2n\,J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2n\, x^{n+2r}}{r!\,\gamma(n+r+1)\, 2^{n+2r}}.$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-2r)}{r!\,\gamma(n+r+1)} \frac{x^{n+2r}}{2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)}{r!\,\gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - \sum_{r=\phi}^{\infty} \frac{(-1)^r\, 2r}{r!\,\gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r)}{r!\,(n+r)\,\gamma(n+r)} \frac{x^{n+2r}}{2^{n+2r}} - \sum_{r=1}^{\infty} \frac{(-1)^r\, 2r}{(r-1)!\, r\, \gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}.$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (x x^{-1})}{r!\,\gamma(n+r)} \cdot \frac{x^{n+2r}}{2^{n+2r-1}} - \sum_{r=1}^{\infty} \frac{(-1)^r (x x^{-1})}{(r-1)!\,\gamma(n+r+1)} \frac{x^{n+2r}}{2^{n+2r-1}}.$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma((n-1)+r+1)} \cdot \underbrace{\frac{x^{n+2r-1}}{2^{n+2r-1}}}_{\left(\frac{x}{2}\right)^{(n-1)+2r}} - x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\gamma(n+r+1)} \cdot \frac{x^{n+2r-1}}{2^{n+2r-1}}.$$

$$= x J_{n-1}(x) - x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\gamma(n+(s+1)+1)} \cdot \frac{x^{n+2(s+1)-1}}{2^{n+2(s+1)-1}}.$$

$[\because r-1 = s \;\&\; r = s+1]$

$$= x J_{n-1}(x) + x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\gamma((n+1)+s+1)} \cdot \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= x J_{n-1}(x) + x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\gamma((n+1)+s+1)} \cdot \left(\frac{x}{2}\right)^{((n+1)+2s)}$$

$$= x J_{n-1}(x) + x J_{n+1}(x).$$

$$\therefore\; 2n\, J_n(x) = x\left\{ J_{n-1}(x) + J_{n+1}(x) \right\}$$

$$J_n(x) = \frac{x}{2n}\left\{ J_{n-1}(x) + J_{n+1}(x) \right\}$$

— x —

Problems on Recurrence identities :-

① Prove that

ⓐ $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ (or) Determine the value of $J_{1/2}(x)$.

ⓑ $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\cos x$ (or) Determine the value of $J_{-\frac{1}{2}}(x)$.

ⓒ $J_{-\frac{1}{2}}(x) = J_{1/2}(x)\cot x$

ⓓ $\left[J_{1/2}(x)\right]^2 + \left[J_{-\frac{1}{2}}(x)\right]^2 = \frac{2}{\pi x}$

ⓐ **Sol** :-

we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$J_n(x) = \left\{\frac{(-1)^0}{0!\,\Gamma(n+1)}\left(\frac{x}{2}\right)^{n+0} + \frac{(-1)^1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^{n+4} + \cdots\right\}$$

$$= \left\{\frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n - \frac{1}{\Gamma(n+2)}\left(\frac{x}{2}\right)^n\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^n\left(\frac{x}{2}\right)^4 + \cdots\right\}$$

$$= \left(\frac{x}{2}\right)^n\left\{\frac{1}{\Gamma(n+1)} - \frac{1}{\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 + \cdots\right\}$$

$\Big[\because\ \Gamma(n+1) = n\Gamma(n) \longrightarrow$ ① $\Gamma(n+2) = \Gamma(\underline{n+1}+1)$
$= (n+1)\,\Gamma(n+1)$

② $\Gamma(n+3) = \Gamma(\underline{n+2}+1) = (n+2)\Gamma(n+2)$
$= (n+2)(n+1)\Gamma(n+1)\Big]$

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\gamma(n+1)} - \frac{1}{(n+1)\gamma(n+1)} \cdot \left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(n+1)(n+2)\,\gamma(n+1)} \left(\frac{x}{2}\right)^4 + \cdots \right\}$$

$$= \frac{1}{\gamma(n+1)} \frac{x^n}{2^n} \left\{ 1 - \frac{1}{2^2 (n+1)} \cdot x^2 + \frac{1}{2^4 \cdot 2!\,(n+1)(n+2)} x^4 - \cdots \right\}$$

$$J_n(x) = \frac{1}{\gamma(n+1)} \cdot \frac{x^n}{2^n} \left\{ 1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 \cdot 2!\,(n+1)(n+2)} x^4 - \cdots \right\}$$

put $n = \frac{1}{2}$,

$$J_{1/2}(x) = \frac{1}{\gamma\left(\frac{1}{2}+1\right)} \cdot \frac{x^{1/2}}{2^{1/2}} \left\{ 1 - \frac{1}{4 \cdot \frac{3}{2}} x^2 + \frac{1}{4^2\, 2!\, \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)} x^4 - \cdots \right\}$$

$$J_{1/2}(x) = \frac{1}{\frac{1}{2}\gamma\left(\frac{1}{2}\right)} \frac{x^{1/2}}{2^{1/2}} \left\{ 1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{4 \cdot 2 \cdot 3 \cdot 5} - \cdots \right\}$$

$$J_{1/2}(x) = \frac{2}{\sqrt{\pi}} \frac{x^{1/2}}{2^{1/2}} \left\{ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right]$$

$\left[\because \gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \;\; \& \;\; \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$

$$J_{1/2}(x) = \frac{2}{\sqrt{\pi}} \cdot \frac{x^{1/2}}{2^{1/2}} \cdot \frac{x}{x} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right]$$

$$= \frac{2 \cdot 2^{-\frac{1}{2}}}{\sqrt{\pi} \cdot x \cdot x^{1/2}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$J_{1/2}(x) = \frac{2^{1/2}}{\sqrt{\pi}\sqrt{x}} \left[(\sin x) \right] = \sqrt{\frac{2}{\pi x}} \sin x$$

— x —

(c) sol :-

we have to prove that

$$J_{-\frac{1}{2}}(x) = J_{1/2}(x)\cot x.$$

we know that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad \& \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$\Rightarrow \quad \frac{J_{-1/2}(x)}{\cos x} = \sqrt{\frac{2}{\pi x}} \quad \& \quad \frac{J_{1/2}(x)}{\sin x} = \sqrt{\frac{2}{\pi x}}$$

$$\Rightarrow \frac{J_{1/2}(x)}{\sin x} = \frac{J_{-\frac{1}{2}}(x)}{\cos x}$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = J_{1/2}(x)\cot x$$

(d) sol :-

$$[J_{1/2}(x)]^2 + [J_{-\frac{1}{2}}(x)]^2 = \frac{2}{\pi x}\sin^2 x + \frac{2}{\pi x}\cos^2 x = \frac{2}{\pi x}$$

$(\because \sin^2 x + \cos^2 x = 1)$.

— x —

(2) prove that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x}\sin x - \cos x\right]$

(or)

Express $J_{3/2}(x)$ in terms of sine & cosine functions.

sol :- we know the recurrence relation

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \longrightarrow (1)$$

substitute $n = \frac{1}{2}$, in ①

$$J_{1/2}(x) = \frac{x}{2 \cdot \frac{1}{2}} \left[J_{\frac{1}{2}-1}(x) + J_{\frac{1}{2}+1}(x) \right]$$

$$J_{1/2}(x) = x \left[J_{-\frac{1}{2}}(x) + J_{3/2}(x) \right]$$

$$J_{1/2}(x) - x J_{-\frac{1}{2}}(x) = x J_{3/2}(x),$$

dividing with x,

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-\frac{1}{2}}(x).$$

$$= \frac{1}{x} \sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left(\frac{1}{x} \sin x - \cos x \right).$$

③ prove that

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{3-x^2}{x^2} \sin x - \frac{3}{x} \cos x \right].$$

(or)

write $J_{5/2}(x)$ in finite form.

<u>Sol</u>:- From recurrence formula

$$J_n(x) = \frac{x}{2n} \left[J_{n+1}(x) + J_{n-1}(x) \right].$$

put $n = \frac{3}{2}$,

$$J_{3/2}(x) = \frac{x}{2 \cdot \frac{3}{2}} \left[J_{\frac{3}{2}+1}(x) + J_{\frac{3}{2}-1}(x) \right]$$

$$3 J_{3/2}(x) = x \left[J_{5/2}(x) + J_{1/2}(x) \right]$$

$$3 J_{3/2}(x) - x J_{1/2}(x) = x J_{5/2}(x),$$ dividing with x

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$$

we know that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) \quad (\because \text{problam } ②)$$

$$J_{5/2}(x) = \frac{3}{x} \cdot \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) - \sqrt{\frac{2}{\pi x}} \sin x.$$

$$= \sqrt{\frac{2}{\pi x}} \left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} \right) - \sqrt{\frac{2}{\pi x}} \sin x$$

$$= \sqrt{\frac{2}{\pi x}} \left(\frac{3}{x^2} - 1 \right) \sin x - \sqrt{\frac{2}{\pi x}} \cdot \frac{3\cos x}{x}$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3 - x^2}{x^2} \right) \sin x - \frac{3}{x} \cos x \right]$$

— x —

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$$

we know that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) \quad (\because \text{ problem } 2)$$

$$J_{5/2}(x) = \frac{3}{x} \cdot \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) - \sqrt{\frac{2}{\pi x}} \sin x.$$

$$= \sqrt{\frac{2}{\pi x}} \left(\frac{3 \sin x}{x^2} - \frac{3 \cos x}{x} \right) - \sqrt{\frac{2}{\pi x}} \sin x$$

$$= \sqrt{\frac{2}{\pi x}} \left(\frac{3}{x^2} - 1 \right) \sin x - \sqrt{\frac{2}{\pi x}} \cdot \frac{3 \cos x}{x}$$

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\left(\frac{3 - x^2}{x^2} \right) \sin x - \frac{3}{x} \cos x \right]$$

— x —

4. Show that (i) $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x} \right) J_1(x) + \left(1 - \frac{24}{x^2} \right) J_0(x)$

(or)

Express $J_4(x)$ in terms of $J_0(x)$ & $J_1(x)$.

Sol :-

From recurrence relation,

$$J_n(x) = \frac{x}{2n} \left[J_{n-1}(x) + J_{n+1}(x) \right]$$

i.e, $$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\Rightarrow J_{n+1}(x) = -J_{n-1}(x) + \frac{2n}{x} J_n(x)$$

$$\Rightarrow J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

putting $n = 1, 2, 3, 4, \ldots$ successively, we obtain

$n=1$, $J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \longrightarrow (1)$

$n=2$, $J_3(x) = \frac{4}{x} J_2(x) - J_1(x) \longrightarrow (2)$

$n=3$, $J_4(x) = \frac{6}{x} J_3(x) - J_2(x). \longrightarrow (3)$

First we find $J_3(x)$,

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

$$= \frac{4}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - J_1(x) \quad [\because (1)]$$

$$= \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x).$$

$$= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x) \longrightarrow (4)$$

Now $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$

$$= \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$J_4(x) = \left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right) J_1(x) - \left(\frac{24}{x^2} + 1\right) J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

— x —

(5) prove that

$$\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$$

Sol :-

$$\frac{d}{dx}\left[x J_n J_{n+1}\right] = J_n J_{n+1} + x J_n' J_{n+1} + x J_n J_{n+1}'$$

$$= J_n J_{n+1} + (x J_n') J_{n+1} + J_n (J_{n+1}' x) \rightarrow (1)$$

we know the recurrence relations

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \longrightarrow (2)$$

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \longrightarrow (3)$$

In eqn (3), put $n = n+1$

$$x J_{n+1}' = -(n+1) J_{n+1} + x J_n \rightarrow (4)$$

substitute (2) & (4) in (1),

$$\frac{d}{dx}\left[x J_n J_{n+1}\right] = J_n J_{n+1} + (n J_n - x J_{n+1}) J_{n+1} + J_n\left(-(n+1) J_{n+1} + x J_n\right)$$

$$= J_n J_{n+1} + n J_n J_{n+1} - x J_{n+1}^2 - (n+1) J_n J_{n+1} + x J_n^2$$

$$= \cancel{J_n J_{n+1}} + \cancel{n J_n J_{n+1}} - x J_{n+1}^2 - \cancel{n J_n J_{n+1}} - \cancel{J_n J_{n+1}} + x J_n^2$$

$$= x\left[J_n^2 - J_{n+1}^2\right]$$

$$\therefore \frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right].$$

—— x ——

(6) Evaluate $\int x^3 J_0(x)\,dx$.

Sol:-

we know that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).$$

on integrating,

$$x^n J_n(x) = \int x^n J_{n-1}(x)\,dx \longrightarrow (1)$$

Put $n=1$ in (1):-

$$\int x J_0(x)\,dx = x J_1(x) \longrightarrow (2) \quad \& \quad \int x^2 J_1(x)\,dx = x^2 J_2(x) \quad (\because n=2). \longrightarrow (3)$$

Now $$\int x^3 J_0(x)\,dx = \int \underset{u}{x^2}\,[\underset{v}{x J_0(x)}]\,dx$$

$$= x^2 \int x J_0(x)\,dx - \int \left[\int x J_0(x)\,dx\right] 2x\,dx$$

$$= x^2 \cdot (x J_1(x)) - 2\int x\,(x J_1(x))\,dx \quad (\because (2))$$

$$= x^3 J_1(x) - 2\int x^2 J_1(x)\,dx$$

$$= x^3 J_1(x) - 2x^2 J_2(x) + C \quad (\because (3))$$

$$\therefore \int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) + C$$

— x —

① prove that $J_{-n}(x) = (-1)^n J_n(x)$ where n is a positive integer. (10)

Sol:- we know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$\therefore J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(-n+r+1)} \cdot \left(\frac{x}{2}\right)^{-n+2r}.$$

we know that the gamma function of zero (or) negative integers is infinite.

$\therefore$ terms in $J_{-n}(x)$ are are equal to zero till $-n+r+1 < 1$ (or) $r < n$.

so we can write

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,\gamma(-n+r+1)} \cdot \left(\frac{x}{2}\right)^{-n+2r}.$$

putting $r = n+s \Rightarrow r-n = s. \Rightarrow s \to 0$ to ∞.

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)!\,\gamma(-n+(n+s)+1)} \left(\frac{x}{2}\right)^{-n+2(n+s)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{\gamma(n+s+1)\; s!} \left(\frac{x}{2}\right)^{n+2s}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\gamma(n+s+1)} \cdot \left(\frac{x}{2}\right)^{n+2s}$$

$[\because \gamma(s+1) = s!$, & $\gamma(n+s+1) = (n+s)!]$

② prove that $J_n(-x) = (-1)^n J_n(x)$, wher n is +ve (or) -ve integer.

(25) Prove that

Sol :- we have $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \longrightarrow (1)$

Case (1) :- If n is a +ve integer.

putting $x = -x$ in (1),

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \gamma(n+r+1)} \left(\frac{-x}{2}\right)^{n+2r}.$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \gamma(n+r+1)} (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r}.$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \left(\because (-1)^{2r} = 1\right)$$

$$= (-1)^n J_n(x).$$

Case (2) :- If n is a negative integer.

Let $n = -m$, where m is a +ve integer.

then $J_n(x) = J_{-m}(x) = (-1)^m J_m(x)$.

put $x = -x$ in above relation,

$$J_n(-x) = (-1)^m J_m(-x). = (-1)^m \left[(-1)^m J_m(x)\right] \quad [\because \text{case (1)}]$$

$$= (-1)^{2m} J_m(x) = J_m(x) = J_{-n}(x)$$

$$= (-1)^n J_n(x)$$

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(7) Evaluate $\int x^3 J_0(x)\,dx$.

Sol :- we know that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x).$$

(or) $\int x^n J_{n-1}(x)\,dx = x^n J_n(x). \longrightarrow ①$

$$\int x^3 J_0(x)\,dx = \int x^2 \left[x J_0(x)\right] dx$$

$$= x^2 \int x J_0(x)\,dx - \int \left[\int x J_0(x)\,dx\right] 2x\,dx$$

$$= x^2 \cdot x J_1(x) - \int 2x \cdot \left(x J_1(x)\right) dx \quad \left[\because n=1 \text{ in } ① \Rightarrow \int x J_0(x)\,dx = x J_1(x)\right]$$

$$= x^3 J_1(x) - 2\int x^2 J_1(x)\,dx.$$

$$\left[\because n=2 \text{ in } ① \Rightarrow \int x^2 J_1(x)\,dx = x^2 J_2(x)\right]$$

$$= x^3 J_1(x) - 2x^2 J_2(x).$$

$$\therefore \int x^3 J_0(x)\,dx = x^3 J_1(x) - 2x^2 J_2(x) + c.$$

(8) show that $\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[n J_n^2 - (n+1) J_{n+1}^2\right]$.

Sol :- we have

$$\frac{d}{dx}\left(J_n^2 + J_{n+1}^2\right) = 2 J_n J_n' + 2 J_{n+1} J_{n+1}' \longrightarrow ①$$

we know the recurrence relation,

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x).$$

(or) $J_n' = \frac{n}{x} J_n(x) - J_{n+1}(x) \longrightarrow ②$

$$J_n' = -\frac{n}{x} J_n + J_{n-1} \longrightarrow ③$$

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substituting $n+1$ in n in ③, replacing

$$J_{n+1}' = -\frac{n+1}{x} J_{n+1} + J_n \longrightarrow ④.$$

substituting ② & ④ in ①, we get

$$\frac{d}{dx}\left(J_n^2 + J_{n+1}^2\right)$$

$$= 2J_n\left[\frac{n}{x}J_n - J_{n+1}\right] + 2J_{n+1}\left[-\frac{n+1}{x}J_{n+1} + J_n\right]$$

$$= \frac{2}{x}\left[nJ_n^2 - nJ_nJ_{n+1} - J_{n+1}^2(n+1) + J_nJ_{n+1}\right]$$

$$= \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right].$$

$$\therefore \frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right].$$

⑨ prove that $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = 1.$

Sol:- we have $\frac{d}{dx}\left(J_n^2 + J_{n+1}^2\right) = \frac{2}{x}\left[nJ_n^2 - (n+1)J_{n+1}^2\right] \longrightarrow ①$

$$= 2\left[\frac{n}{x}J_n^2 - \frac{(n+1)}{x}J_{n+1}^2\right].$$

putting $n = 0, 1, 2, \cdots$ in ①, we have

$$\frac{d}{dx}\left(J_0^2 + J_1^2\right) = 2\left[0 - \frac{1}{x}J_1^2\right]$$

$$\frac{d}{dx}\left(J_1^2 + J_2^2\right) = 2\left[\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right]$$

$$\frac{d}{dx}\left(J_2^2 + J_3^2\right) = 2\left[\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right]$$

$$\cdots\cdots\cdots$$

Adding these column wise and remembering that $J_n(x) \to 0$ as $n \to \infty$, we get

$$\frac{d}{dx}\left\{J_0^2 + 2\left(J_1^2 + J_2^2 + J_3^2 + \cdots\right)\right\} = 0$$

Integrating w.r.to x, we get $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = c$, $\longrightarrow ②$

where c is constant.

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Also we know that

$J_0(0) = 1$ & $J_n(0) = 0$ for $n \geq 1$

$\therefore$ From (2), putting $x = 0$, we have $1 + 2(0) = c$

$\Rightarrow c = 1.$

$$\therefore J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = 1.$$

(10) Show that $\int J_3(x)\,dx = -J_2(x) - \frac{2}{x} J_1(x)$.

<u>Sol</u>:- we have the recurrence relation,

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x).$$

by integrating w.r.to x, we get

$$x^{-n} J_n(x) = \int -x^{-n} J_{n+1}(x)\,dx.$$

$$\therefore \int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x). \longrightarrow (1)$$

$$\int a b = a\int b - \int a' \int b.$$

Now $\int J_3(x)\,dx$

$$= \int \overset{a}{x^2} \overset{b}{\left[x^{-2} J_3(x)\right]}\,dx = x^2\left[\int x^{-2} J_3(x)\,dx\right] - \int 2x \left[\int x^{-2} J_2(x)\right]dx$$

$$= x^2\left[-x^{-2} J_2(x)\right] + \int \left[2x \cdot x^{-2} J_2(x)\right]dx$$

$\left[\because \text{ by putting } n = 2, \text{ in } (1), \int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x)\right]$

$$= -J_2(x) + \int \frac{2}{x} J_2(x)\,dx.$$

For $n = 1$, $-x^{-1} J_1(x) = \int x^{-1} J_2(x)\,dx.$

$$\therefore \int J_3(x)\,dx = -J_2(x) + 2\cdot\left(-x^{-1} J_1(x)\right)$$

$$= -J_2(x) - \frac{2}{x} J_1(x)$$

$$\int J_3(x)\,dx + J_2(x) - \frac{2}{x} J_1(x) = 0.$$

Generating function for $J_n(x)$:—

$$\boxed{e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x).}$$

(or)

Show that the coefficient of t^n in the power series expansion of $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ is $J_n(x)$.

Sol :-

we know that

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^6}{6!}+\cdots$$

$$L.H.S = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$$

$$= e^{\frac{xt}{2}-\frac{x}{2t}} = e^{\frac{xt}{2}} \cdot e^{-\frac{x}{2t}}$$

$$= \left[1+\left(\frac{xt}{2}\right)+\frac{1}{2!}\left(\frac{xt}{2}\right)^2+\frac{1}{3!}\left(\frac{xt}{2}\right)^3+\cdots\cdots+\frac{1}{n!}\left(\frac{xt}{2}\right)^n+\frac{1}{(n+1)!}\left(\frac{xt}{2}\right)^{n+1}+\cdots\right]\times$$

$$\left[1-\left(\frac{x}{2t}\right)+\frac{1}{2!}\left(\frac{x}{2t}\right)^2-\cdots+\frac{(-1)^n}{n!}\left(\frac{x}{2t}\right)^n+\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2t}\right)^{n+1}+\cdots\right]$$

$$= \left[t^0+\left(\frac{x}{2}\right)t+\frac{1}{2!}\left(\frac{x}{2}\right)^2 t^2+\cdots\cdots+\frac{1}{n!}\left(\frac{x}{2}\right)^n t^n+\frac{1}{(n+1)!}\cdot\left(\frac{x}{2}\right)^{n+1} t^{n+1}+\cdots\right].$$

$$\left[t^0-\left(\frac{x}{2}\right)t^{-1}+\frac{1}{2!}\left(\frac{x}{2}\right)^2 t^{-2}+\cdots\cdots+\frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n t^{-n} + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+1} t^{-(n+1)}+\cdots\right]$$

The coefficient of t^n in this product
= sum of the products of the coefficients of $t^n, t^{n+1}, t^{n+2}, \ldots$ in the first bracket with the coefficients of $t^0, t^{-1}, t^{-2}, \ldots$

in the second bracket.

$\therefore$ coefficient of t^n

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n + \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+1}\left(-\frac{x}{2}\right) + \frac{1}{(n+2)!}\left(\frac{x}{2}\right)^{n+2}\left\{\frac{1}{2!}\left(\frac{x}{2}\right)^2\right\}$$

$$+ \cdots\cdots$$

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{(n+2)!\,2!}\left(\frac{x}{2}\right)^{n+4} - \cdots\cdots$$

$$= \frac{(-1)^0}{0!\,\gamma(n+1)}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!\,\gamma(n+2)}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\gamma(n+3)}\left(\frac{x}{2}\right)^{n+4} + \cdots$$

$$[\because \gamma(n+1) = n!]$$

$$= \frac{(-1)^0}{0!\,\gamma(n+1)}\left(\frac{x}{2}\right)^{n+2(0)} + \frac{(-1)^1}{1!\,\gamma(n+1+1)}\cdot\left(\frac{x}{2}\right)^{n+2(1)} + \frac{(-1)^2}{2!\,\gamma(n+2+1)}\left(\frac{x}{2}\right)^{n+2(2)}$$

$$+ \cdots\cdots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\gamma(n+r+1)}\cdot\left(\frac{x}{2}\right)^{n+2r} = J_n(x).$$

In a similar manner the coefficient of t^{-n} in the product

= sum of products of the coefficient of t^{-n}, $t^{-(n+1)}$, $t^{-(n+2)}$, ... in the second bracket with the coefficients of t^0, t^1, t^2, ... in the first bracket.

coefficient of t^{-n}

$$= \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+1}\cdot\left(\frac{x}{2}\right) + \frac{(-1)^{n+2}}{(n+2)!}\left(\frac{x}{2}\right)^{n+2}\left\{\frac{1}{2!}\left(\frac{x}{2}\right)^2\right\} + \cdots$$

$$= (-1)^n\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n + \frac{-1}{(n+1)!}\cdot\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \cdots\right]$$

$$= (-1)^n\left[\frac{(-1)^0}{0!\,\gamma(n+1)}\left(\frac{x}{2}\right)^n + \frac{(-1)}{\gamma(n+2)}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\gamma(n+3)}\cdot\left(\frac{x}{2}\right)^{n+4} + \cdots\right]$$

$$= (-1)^n\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}\right] = (-1)^n J_n(x).$$

Finally the coefficient of t^0 in the product = sum of the products of the coefficients of $t^0, t^1, t^2, \ldots$ in the first bracket with the coefficient of $t^0, t^{-1}, t^{-2}, t^{-3}, \ldots$ in the second bracket.

coefficient of t^0

$$= 1\cdot 1 + \left(\frac{x}{2}\right)\left(-\frac{x}{2}\right) + \left\{\frac{1}{2!}\left(\frac{x}{2}\right)^2\right\}\left\{\frac{1}{2!}\left(\frac{x}{2}\right)^2\right\} + \cdots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

$$= J_0(x).$$

Hence $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$

$$= J_0 + tJ_1(x) + t^2 J_2(x) + \cdots + t^n J_n(x) + \cdots$$
$$+ t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + \cdots + t^{-n}J_{-n}(x) + \cdots$$

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$$= J_0(x) + t J_1(x) + t^2 J_2(x) + \cdots + t^n J_n(x) + \cdots$$
$$+ t^{-1}(-J_1(x)) + t^{-2} J_2(x) + \cdots + t^{-n}(-1)^n J_n(x) + \cdots$$
$$[\because J_{-n}(x) = (-1)^n J_n(x)].$$

$$= J_0 + (t - t^{-1}) J_1(x) + (t^2 + t^{-2}) J_2(x) + \cdots$$
$$\cdots + [t^n + (-1)^n t^{-n}] J_n(x) + \cdots$$

$$= J_0 + \left(t - \frac{1}{t}\right) J_1(x) + \left(t^2 + \frac{1}{t^2}\right) J_2(x) + \cdots$$
$$\cdots + \left[t^n + \frac{(-1)^n}{t^n}\right] J_n(x) + \cdots$$

$$= \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

This shows that Bessel functions of various orders can be defined as coefficient of different powers of t in the expansions of $e^{\frac{x}{2}(t - \frac{1}{t})}$.

For this reason, it is known as generating function of Bessel function

—x—

Problems on generating function :-

1. show that

(a) $\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots\cdots)$

(b) $\sin(x\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + J_5\sin 5\theta + \cdots\cdots)$

Sol:- we know the generating function of $J_n(x)$ is given by

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

$$= J_0 + J_1\left(t-\frac{1}{t}\right) + J_2\left(t^2+\frac{1}{t^2}\right) + J_3\left(t^3-\frac{1}{t^3}\right) + \cdots\cdots \quad (1)$$

$$\left[\because J_{-n}(x) = (-1)^n J_n(x)\right]$$

Now put $t = \cos\theta + i\sin\theta$ so that

$t^p = (\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta \longrightarrow (2)$

$[\because (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$.

$t^{-p} = (\cos\theta + i\sin\theta)^{-p} = \cos p\theta - i\sin p\theta$

$\Rightarrow \frac{1}{t^p} = \cos p\theta - i\sin p\theta. \longrightarrow (3)$

$(2) + (3) \Rightarrow t^p + \frac{1}{t^p} = 2\cos p\theta$

$(2) - (3) \Rightarrow t^p - \frac{1}{t^p} = 2i\sin p\theta.$

substituting $p = 1, 2, 3, \cdots\cdots$

$t + \frac{1}{t} = 2\cos\theta, \quad t^2 + \frac{1}{t^2} = 2\cos 2\theta, \cdots\cdots$

$t - \frac{1}{t} = 2i\sin\theta, \quad t^2 - \frac{1}{t^2} = 2i\sin 2\theta, \cdots\cdots$

substituting these values in eqn ① &

$t = \cos\theta + i\sin\theta$ & $\frac{1}{t} = \cos\theta - i\sin\theta$ we get

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = e^{\frac{x}{2}(2i\sin\theta)}$$

$$= e^{ix\sin\theta}.$$

From eqn ①,

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = e^{ix\sin\theta}$$

$$= J_0 + J_1\left(t - \frac{1}{t}\right) + J_2\left(t^2 + \frac{1}{t^2}\right) + J_3\left(t^3 - \frac{1}{t^3}\right) + \cdots$$

$$= J_0 + J_1(2i\sin\theta) + J_2(2\cos 2\theta) + J_3(2i\sin 3\theta) + \cdots$$

$$e^{ix\sin\theta} = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots) + 2i(J_1\sin\theta + J_3\sin 3\theta + \cdots)$$

$$\therefore\ e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta) \qquad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$= J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots) + 2i(J_1\sin\theta + J_3\sin 3\theta + \cdots)$$

equating real & imaginary parts

$$\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots)$$

$$\sin(x\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + \cdots)$$

— x —

(3) Show that

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \left(\cos(n\theta - x\sin\theta)\right) d\theta,$$

n being an integer.

sol :-

$\cos(A-B) = \cos A\cos B + \sin A\sin B.$

$\cos(n\theta - x\sin\theta) = \cos n\theta\cos(x\sin\theta) + \sin n\theta\sin(x\sin\theta).$

From Jacobi series, we know that

$$\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots) \rightarrow (1)$$

and $$\sin(x\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + \cdots) \rightarrow (2)$$

Also we know that

$$\int_0^{\pi}\sin m\theta\sin n\theta\, d\theta = \int_0^{\pi}\cos m\theta\cos n\theta\, d\theta = \begin{cases}\frac{\pi}{2} & \text{if } m=n\\ 0 & \text{if } m\neq n\end{cases} \rightarrow (3)$$

multiplying (1) by $\cos n\theta$ and integrating w.r.to θ from 0 to π,

$$\int_0^{\pi}\cos n\theta\cos(x\sin\theta)\, d\theta = \int_0^{\pi}\cos n\theta\left[J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots\right] d\theta$$

$$= \int_0^\pi J_0 \cos n\theta + 2\int_0^\pi J_2 \cos 2\theta \cos n\theta \, d\theta + 2\int_0^\pi J_4 \cos 4\theta \cos n\theta \, d\theta + \cdots$$

$$= J_0\left(\frac{\sin n\theta}{n}\right)_0^\pi + 2\int_0^\pi J_2 \cos 2\theta \cos n\theta \, d\theta + 2\int_0^\pi J_4 \cos 4\theta \cos n\theta \, d\theta + \cdots$$

$$= 0 + \left\{ 2\int_0^\pi J_m \cos m\theta \cos n\theta \, d\theta \right\}$$ [$\because$ Here m is even]

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2 \cdot J_m \cdot \left(\frac{\pi}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$ [$\because$ formula 3] ($\because$ if n is even we have $m=n$).

$$= \begin{cases} 0 & \text{if } n \text{ is odd } (m \neq n) \\ J_m \cdot \pi & \text{if } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \pi J_n & \text{if } n \text{ is even.} \end{cases} \quad (\because m=n) \longrightarrow (4)$$

Now multiplying (2) with $\sin n\theta$ and integrating w.r.to θ from 0 to π,

$$\int_0^\pi \sin(x \sin\theta) \sin n\theta \, d\theta$$

$$= \int_0^\pi \sin n\theta \left[2(J_1 \sin\theta + J_3 \sin 3\theta + \cdots) \right] d\theta$$

$$= \int_0^{\pi} 2 J_1 \sin\theta \sin n\theta \, d\theta + \int_0^{\pi} 2 \cdot J_3 \sin 3\theta \sin n\theta \, d\theta + \cdots$$

$$= \int_0^{\pi} 2 J_m \sin m\theta \sin n\theta \, d\theta$$ (Here m is odd). and we say $\boxed{m=n}$ if n is odd.

$$= \begin{cases} 2 \cdot J_m \left(\frac{\pi}{2}\right) & \text{if } n \text{ is odd } (m=n) \\ 0 & \text{if } n \text{ is even } (\because m \neq n). \end{cases}$$

$$= \begin{cases} J_m \cdot \pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$= \begin{cases} J_n \cdot \pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \longrightarrow (5)$$

Now we have 2 cases.

Case (1) :- If n is even.

From (4) & (5),

$$(4) + (5) \Rightarrow \int_0^{\pi} [\cos n\theta \cos(x \sin\theta) + \sin n\theta \sin(x \sin\theta)] \, d\theta$$

$$= \pi \cdot J_n(x) + 0 = \pi J_n(x).$$

$$\therefore J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x\sin\theta) \, d\theta.$$

Case (2) :- If n is odd.

From (4) & (5),

$$\int_0^{\pi} [\cos n\theta \cos(x\sin\theta) + \sin n\theta \sin(x\sin\theta)]\, d\theta$$

$$= 0 + J_n \pi .$$

$$\therefore \int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta = J_n(x) \cdot \pi$$

$$\Rightarrow \quad J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta .$$

In general if n is the integer

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta .$$

— x —

(4) using Jacobi series, prove that

$$J_0^2 + 2[J_1^2 + J_2^2 + J_3^2 + \cdots] = 1$$

Sol:- we know the Jacobi series are

$$\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4 \cos 4\theta + \cdots\cdots) \rightarrow (1)$$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \cdots\cdots) \rightarrow (2)$$

Also we know that

$$\int_0^{\pi} \sin m\theta \sin n\theta\, d\theta = \int_0^{\pi} \cos m\theta \cos n\theta\, d\theta = \begin{cases} \frac{\pi}{2}, & \text{if } m=n \\ 0, & \text{if } m\neq n. \end{cases}$$

squaring ① on both sides & integrating w.r.to θ from 0 to π

$$\int_0^{\pi} \cos^2(x\sin\theta)\,d\theta = J_0^2\int_0^{\pi} d\theta + 2^2\int_0^{\pi}\left(J_2^2\cos^2 2\theta + J_4^2\cos^2 4\theta + \cdots\right)d\theta$$

$$= J_0^2\,(\theta)_0^{\pi} + 2^2\int_0^{\pi} J_2^2\cos^2 2\theta\,d\theta + 2^2\int_0^{\pi} J_4^2\cos^2 4\theta\,d\theta + \cdots$$

$$= J_0^2(\pi - 0) + 2^2 J_2^2\left(\frac{\pi}{2}\right) + 2^2 J_4^2\left(\frac{\pi}{2}\right) + \cdots$$

$$= J_0^2\pi + 2J_2^2\pi + 2J_4^2\pi + \cdots \longrightarrow ③$$

$\left[\because \int_0^{\pi}\cos^2 2\theta\,d\theta = \int_0^{\pi}\cos 2\theta\cos 2\theta\,d\theta = \frac{\pi}{2}\ (m=n).\right.$

$|||^{ly}$ $\int_0^{\pi}\sin^2(x\sin\theta)\,d\theta$

$$= 2J_1^2\pi + 2J_3^2\pi + 2J_5^2\pi + \cdots \quad (\because ②).$$

$\longrightarrow ④$

$$③ + ④ \Rightarrow \int_0^{\pi}\left[\cos^2(x\sin\theta) + \sin^2(x\sin\theta)\right]d\theta = \pi\left[J_0^2 + 2\left(J_1^2 + J_2^2 + J_3^2 + \cdots\right)\right] = 1$$

$$\Rightarrow (\theta)_0^{\pi} = \pi\left[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots)\right] = 1$$

$(\because$ in L.H.S, $\sin^2\theta + \cos^2\theta = 1)$

$$\Rightarrow \cancel{\pi} = \cancel{\pi}\left[J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots)\right] = 1$$

$$\Rightarrow J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = 1$$

— x —

Orthogonality of Bessel functions :-

Statement :- If α & β are two distinct roots of $J_n(x) = 0$ then

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2 & \text{if } \alpha = \beta . \end{cases}$$

Proof :-

we know that $J_n(\lambda x)$ is the solution of the eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0 \longrightarrow (1)$$

If $u = J_n(\alpha x)$ & $v = J_n(\beta x)$ are solutions of (1)

then $$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (\alpha^2x^2 - n^2)u = 0 \longrightarrow (2)$$

and $$x^2 \frac{d^2v}{dx^2} + x\frac{dv}{dx} + (\beta^2x^2 - n^2)v = 0 \longrightarrow (3)$$

multiplying (2) by $\frac{v}{x}$ & (3) by $\frac{u}{x}$ and subtracting

$$(2)\frac{v}{x} - (3)\frac{u}{x} = 0.$$

$$\Rightarrow \frac{v}{x}\left[x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (\alpha^2x^2 - n^2)u\right] - \frac{u}{x}\left[x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx} + (\beta^2x^2 - n^2)v\right] = 0$$

$$\Rightarrow vx\frac{d^2u}{dx^2} + v\frac{du}{dx} + \frac{uv}{x}(\alpha^2x^2-n^2) - \frac{u}{1}x\frac{d^2v}{dx^2} - u\frac{dv}{dx} - \frac{uv}{x}(\beta^2x^2-n^2) = 0$$

$$\Rightarrow x\left[v\frac{d^2u}{dx^2} - u\frac{d^2v}{dx^2}\right] + \left[v\frac{du}{dx} - u\frac{dv}{dx}\right] + \frac{uv}{x}\left[\alpha^2x^2 - \beta^2x^2 - n^2 + n^2\right] = 0$$

$uv' + vu'$

product rule.

$u \quad v$

$$\Rightarrow \frac{d}{dx}\left[x(vu' - uv')\right] + \frac{uv}{x}\left[\alpha^2x^2 - \beta^2x^2\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[x(vu' - uv')\right] + uvx\left[\alpha^2 - \beta^2\right] = 0.$$

$$\Rightarrow \frac{d}{dx}\left[x(vu' - uv')\right] = -uvx(\alpha^2 - \beta^2)$$

$$\Rightarrow \frac{d}{dx}\left[x(vu' - uv')\right] = uvx(\beta^2 - \alpha^2).$$

integrating both sides w.r.to x from 0 to 1 we get

$$\int_0^1 (\beta^2 - \alpha^2)\,uvx\,dx = \int_0^1 \frac{d}{dx}\left[x(vu' - uv')\right]dx$$

$$(\beta^2 - \alpha^2)\int_0^1 x\,uv\,dx = \left\{x\left[vu' - uv'\right]\right\}_0^1$$

$$\int_0^1 x\,J_n(\alpha x)J_n(\beta x)\,dx = \frac{1}{\beta^2 - \alpha^2}\left\{x\left[v\,\alpha J_n'(\alpha x)J_n(\beta x) - J_n(\alpha x)\,\beta J_n'(\beta x)\right]\right\}_0^1$$

$$\left[\because u = J_n(\alpha x) \Rightarrow u' = \alpha J_n'(\alpha x)\ \&\ v = J_n(\beta x) \Rightarrow v' = \beta J_n'(\beta x)\right]$$

$$\therefore \int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx$$

$$= \frac{1}{\beta^2-\alpha^2}\left\{ \alpha J_n'(\alpha) J_n(\beta) - \beta J_n(\alpha) J_n'(\beta)\right\} \rightarrow (4)$$

Case (1) :- If $\alpha \neq \beta$

i.e, α & β distinct roots of $J_n(x)$.

$\Rightarrow J_n(\alpha) = 0$ & $J_n(\beta) = 0$.

Then eqn (4) becomes

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0 \quad \text{if } \alpha \neq \beta.$$

This is known as orthogonality relation of Bessel functions.

Case (2) :- If $\alpha = \beta$

The R.H.S of (4) is of $\frac{0}{0}$ form.

(Here we use L-Hospital's rule).

Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching α.

Then (4), gives

$$\underset{\beta \to \alpha}{Lt} \int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \underset{\beta \to \alpha}{Lt} \frac{\alpha J_n'(\alpha) J_n(\beta) - 0}{\beta^2 - \alpha^2}$$

$(\because \alpha$ constant $J_n(\alpha) = 0)$

$$\int_0^1 x [J_n(\alpha x)]^2\, dx = \underset{\beta \to \alpha}{Lt} \frac{\alpha J_n'(\alpha) J_n'(\beta)}{2\beta} \quad \left[\because \alpha \text{ constant},\ \beta - \text{variable}\right]$$

$$= \frac{\beta J_n'(\beta) J_n'(\beta)}{2\beta} = \frac{1}{2} [J_n'(\beta)]^2$$

[$\because$ From recurrence relation,

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$$\Rightarrow J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha) \quad [\because J_n(\alpha) = 0]$$

$$= -J_{n+1}(\alpha)]$$

$$\therefore \int_0^1 x [J_n(\alpha x)]^2\, dx = \frac{1}{2} [J_{n+1}(\beta)]^2$$

$$\therefore \int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \frac{1}{2} [J_{n+1}(\alpha)]^2, \text{ if } \alpha = \beta.$$

——— X ———

(1) show that $J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta)\, d\theta$

$$= \frac{1}{\pi} \int_0^\pi \cos(x \cos\theta)\, d\theta.$$

(2) we know that

$\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots\cdots). \longrightarrow (1)$

integrating w.r.to θ, both sides of (1)., from 0 to π.

we obtain

$$\int_0^\pi \cos(x\sin\theta)\,d\theta = J_0\int_0^\pi d\theta + 2\int_0^\pi J_2\cos 2\theta\,d\theta + 2\int_0^\pi J_4\cos 4\theta\,d\theta + \cdots$$

$$= J_0\left[\theta\right]_0^\pi + 2J_2\left[\frac{\sin 2\theta}{2}\right]_0^\pi + 2J_4\left[\frac{\sin 4\theta}{4}\right]_0^\pi + \cdots$$

$$= \pi J_0.$$

$$\therefore J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\,d\theta. \longrightarrow (2)$$

Replacing θ by $\frac{\pi}{2}-\theta$ in (1),

$$\cos\left(x\sin\left(\tfrac{\pi}{2}-\theta\right)\right) = J_0 + 2\left(J_2\cos 2\left(\tfrac{\pi}{2}-\theta\right) + J_4\cos 4\left(\tfrac{\pi}{2}-\theta\right) + \cdots\right)$$

$$= J_0 + 2J_2\cos(\pi-2\theta) + J_4\cos(2\pi-4\theta) + \cdots$$

$$= J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta \cdots\cdots$$

Integrating both sides w.r.to θ, from 0 to π

$$\int_0^\pi \cos(x\cos\theta)\,d\theta = \int_0^\pi J_0\,d\theta - 2J_2\int_0^\pi \cos 2\theta\,d\theta + 2J_4\int_0^\pi \cos 4\theta\,d\theta \cdots$$

$$= \pi J_0 - 0 = \pi J_0.$$

$$J_0 = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\,d\theta. \quad (3)$$

$$\therefore (2) \,\&\, (3), \quad J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\,d\theta.$$

(2) Establish the Jacobi series

(i) $\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots$

(ii) $\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \cdots]$

Sol :-

we know that

$\cancel{\cos(x\sin\theta) = J_0 - 2J_2}$

$\cos(x\sin\theta) = J_0 + 2(J_2\cos 2\theta + J_4\cos 4\theta + \cdots) \rightarrow (1)$

$\sin(x\sin\theta) = 2(J_1\sin\theta + J_3\sin 3\theta + \cdots) \rightarrow (2)$

Replacing θ by $\frac{\pi}{2} - \theta$ in (1)

$$\cos\left(x\sin\left(\frac{\pi}{2}-\theta\right)\right) = J_0 + 2\left[J_2\left(\cos 2\left(\frac{\pi}{2}-\theta\right)\right) + 2J_4\left[\left(\cos 4\left(\frac{\pi}{2}-\theta\right)\right)\right] + \cdots\right.$$

$$\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots$$

Replacing θ by $\frac{\pi}{2} - \theta$ in (2),

$$\sin\left(x\sin\left(\frac{\pi}{2}-\theta\right)\right) = 2\left[J_1\sin\left(\frac{\pi}{2}-\theta\right) + J_3\sin 3\left(\frac{\pi}{2}-\theta\right) + \cdots\right]$$

$$\sin(x\cos\theta) = 2\left[J_1\cos\theta + J_3\sin\left(\frac{3\pi}{2} - 3\theta\right) + \cdots\right]$$

$$\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \cdots]$$

— x —

19/08/19

MODULE – III

Complex Functions – Differentiation and Integration

Functions of a complex variable :-

Def: Suppose D is a set of complex numbers. A rule f defined on D which assigns to every z in D, a complex number w, is called a function 'f' or mapping 'f' on D. and we write $w = f(z)$

Here z is a complex variable and can be written as $z = x+iy$ where x, y are real and $i^2 = -1$. The set D is called domain of definition of 'f'. The set of all $w = f(z)$ where $z \in D$ is called the range of 'f'.

The image of z under the function 'f' is $w = f(z)$ and as stated, this is also a complex number and we write $w = f(z) = u+iv$ where u, v are real.

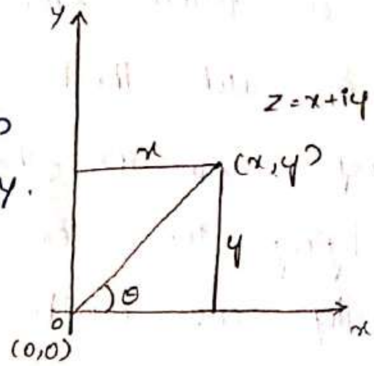
Since $z = x+iy$ and z depends on x and y, we notice that u and v also depend on x and y so that $u = u(x,y)$ and $v = v(x,y)$ hence $w = f(z) = u(x,y) + i\,v(x,y)$ where u is real part of $f(z)$ and v is an imaginary part of $f(z)$.

Geometrical representation of a complex number :-

If $z = x+iy$ be a complex number in the complex plane, then the corresponding order pair (x,y) is a point in the cartesian Geometric plane.

Note:-

1) If $z=x+iy$ be a complex number then the complex conjugate of z is $\bar{z}=x-iy$.



$z=x+iy$

Argand diagram

Polar form of a complex number:-

The point $p(x,y)$ represents a complex number $z=x+iy$ in z plane. Let $x=r\cos\theta$, $y=r\sin\theta$

$z=x+iy$

$z=r\cos\theta+i\cdot r\sin\theta$

$z=r(\cos\theta+i\sin\theta)$

$\boxed{z=re^{i\theta}}$

which is polar form of z.

Modulus of a complex number:-

Let $z=x+iy$ be a complex number then the modulus of z is denoted by $|z|$ and defined as $|z|=\sqrt{x^2+y^2}$

we have $x=r\cos\theta$, $y=r\sin\theta$

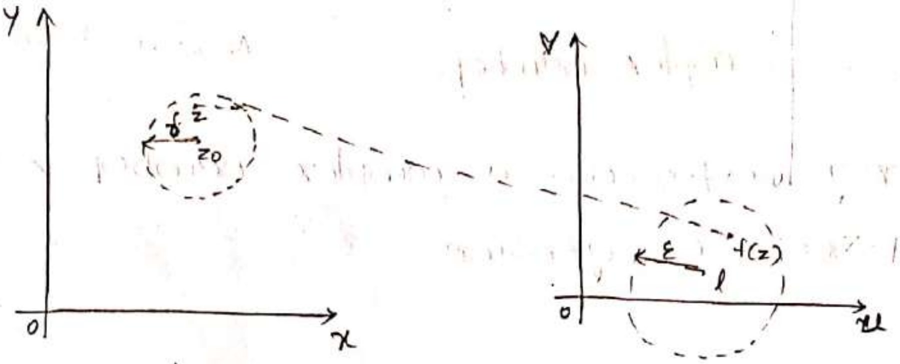
So, $\frac{y}{x}=\tan\theta$

$\theta=\tan^{-1}\left(\frac{y}{x}\right)$ is called amplitude or the argument of a complex number.

Limit and continuity:-

Limit of a function $f(z)$:-

A function $w=f(z)$ is said to tend to limit l as z approaches a point z_0, if for every real ε, we can find a positive δ such that $|f(z)-l| < \varepsilon$ for $0<|z-z_0|<\delta$ i.e., for every $z \neq z_0$ in the δ-disc in z-plane, $f(z)$ as a value lying in the ε-disc of w plane, in this case symbolically we write $\boxed{\lim_{z\to z_0} f(z) = l}$



Note :-

1) If the limit of a function exist as $z \to z_0$ then it is unique

2) Let $f(z) = u(x,y) + iv(x,y)$

where $z = x+iy$, $z_0 = x_0+iy_0$

then the condition that the limit of z exists at zero.

$\lim f(z) = u_0 + iv_0$ is satisfied if and only if $\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y) = u_0$

and $\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) = v_0$

continuity :-

A function $f(z)$ is said to be continuous at $z=z_0$, if $f(z_0)$ is defined and $\lim_{z\to z_0} f(z) = f(z_0)$.

We observe that for the function $f(z)$ to be continuous at z_0 the function must be defined in some neighbourhood of z_0 including z_0 and $\lim_{z \to z_0} f(z) = f(z_0)$.

$f(z)$ is said to be continuous in a domain if it is continuous at every point of that domain.

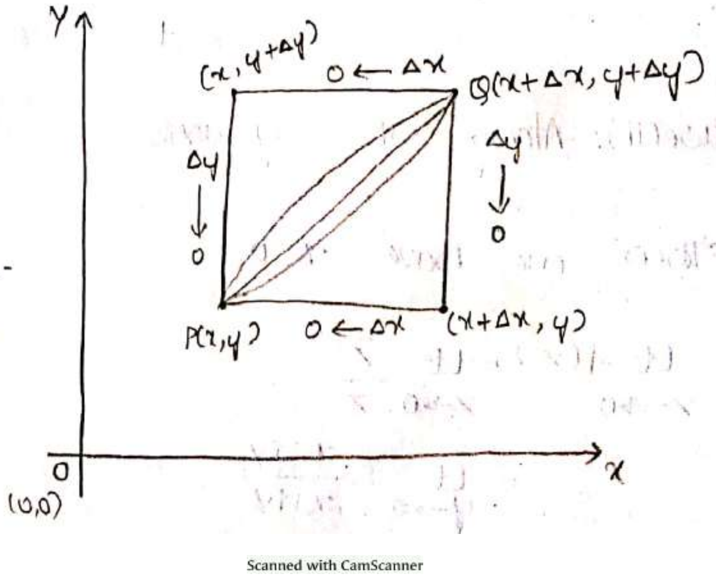
Point of discontinuity:-

If $f(z)$ is not continuous at $z = z_0$ then that point $z = z_0$ is called a point of discontinuity of $f(z)$. For example $z=0$ is the point of discontinuity of the function $f(z) = \frac{1}{z}$

Derivative of a function:-

Let $w = f(z)$ be a given function is single valid function of the variable $z = x+iy$ then the derivative of $w=f(z)$ is defined to be $\frac{dw}{dz} = f'(z) = \underset{z \to 0}{Lt} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ provided the limit exist. and has the same value for all the different ways in which Δz approaches to zero.

Suppose $P(z)$ is a fixed point and $Q(z+\Delta z)$ is a neighbourhood point the point Q may approaches P along any straight line or curved path in the given region i.e., $\Delta z \to 0$ in any manner.



Note:-

1) If a function is differentiable at a point then it is continuous there.

2. But converse need not be true.

Problems:-

1) Show that the function $f(z) = \frac{\bar{z}}{z}$ is not continuous at $z=0$.

Sol^n: $f(z) = \frac{\bar{z}}{z}$

$z = x+iy$ then $\bar{z} = x-iy$

$f(z) = \frac{\bar{z}}{z} = \frac{x-iy}{x+iy}$

If $z = x$ then $\bar{z} = x$ and

If $z = iy$ then $\bar{z} = -iy$

Case(i): Along x-axis:-

Then we have $y=0$

So, $f(z) = \frac{\bar{z}}{z} = \frac{x}{x}$

$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\bar{z}}{z}$ and

$$\lim_{x\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\bar{z}}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x-iy}{x+iy}$$

$$= \lim_{x\to 0} \left(\frac{x}{x}\right)$$

$$= 1$$

Case(ii): Along the y-axis

Then we have $x=0$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\bar{z}}{z}$$

$$= \lim_{\substack{y\to 0\\ x\to 0}} \frac{x-iy}{x+iy}$$

$$= \lim_{y \to 0} \left(\frac{-iy}{iy} \right)$$

$$= -1$$

$\therefore \lim_{z \to 0} f(z) = -1$

$\therefore \lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\bar{z}}{z}$ doesnot exist

$\therefore f(z)$ is not continuous at $z=0$

Analytic function (Regular Function):-

Let a function $f(z)$ be derivable at every point z in an ε neighbourhood of z_0 i.e., $f'(z)$ exist $\forall z$ such that $|z-z_0| < \varepsilon$ where $\varepsilon > 0$ then $f(z)$ is said to be analytic at z_0.

Note:-

If $f(z)$ is analytic at z_0,

1) $f'(z_0)$ exist and

2) $f'(z)$ exist at every point z in a neighbourhood of z_0.

Entire function (Integral function):-

Let 'D' be a domain of complex numbers.

If $f(z)$ is analytic at every point $z \in D$, $f(z)$ is said to be analytic in the domain 'D' and if $f(z)$ is analytic at every point on the complex plane, $f(z)$ is said to be an entire function (or) Integral function.

Singular point:-

If $f'(z_0)$ doesnot exist at the point $z=z_0$ then $z=z_0$ is called singular point of the function $f(z)$. i.e, at which the singular point $z=z_0$ the function $f(z)$ is not analytic.

Isolated singular point:-

If $f'(z)$ exist at every point in the neighbourhood of z_0 but $f'(z_0)$ doesnot exist, then z_0 is said to be an isolated singular point of $f(z)$.

For example, $f(z)=\frac{1}{z}$ is analytic at every point, $z\neq0$

$\Rightarrow f'(z) = -\frac{1}{z^2}$, if $z\neq0$

At $z=0$ $f'(z)$ doesnot exist. so $z=0$ is an isolated singular point of $f(z)$.

Cauchy's - Riemann equations (C-R):-

[~~Theorem:-~~]

The necessary and sufficient condition for the derivative of the function $f(z)=w=u(x,y)+iv(x,y)$ to exist for all values of z in domain R. are

1) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in 'R'.

2) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ (i.e., $u_x=v_y$, $u_y=-v_x$)

The above relations are known as cauchy's Riemann eq^{n}s

Analytic function :– A function $w = f(z)$ is said to be analytical in a domain 'G' if $f(z)$ is differentiable at every point of 'G'. i.e., $f'(z)$ exists. It is also known as a "Regular" (or) "holomorphic" function.

× Regular Function :– If $f(z)$ is analytic at every 'z' on the whole complex plane, then $f(z)$ is said to be "Regular function". ×

Entire function :– A function which is analytic everywhere in the complex plane, is known as "Entire function." & "integral function."

PROPERTIES OF ANALYTIC FUNCTIONS :–

(1) If $f(z)$ and $g(z)$ are analytic functions then $f \pm g$, $f.g$ and $\frac{f}{g}$ are also analytic functions provided $g \neq 0$.

(2) Analytic function of an analytic function is analytic.

(3) An entire function of an entire function is entire.

(4) Derivative of an analytic function is itself analytic.

Singular point :– A point at which an analytic function ceases to possess a derivative is called a "Singular point" of the function.

CAUCHY - RIEMANN (C-R) EQUATIONS :-

The necessary and Sufficient Conditions that the functions $w = f(z) = u(x,y) + i\,v(x,y)$ to be analytic in a region 'R' are

(a) the four first order derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous in R.

(b) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. (C-R Equations)

Proof:- Necessary Condition:-

Let $w = f(z) = u(x,y) + i\,v(x,y)$ be analytic in a region R. $\longrightarrow ①$

We have to prove that the C-R Equations are satisfied.

Given, $f(z) = u + iv$. where $z = x+iy$

Let 'δx', 'δy' be the increments in 'x' and 'y' respectively.

Let 'δu', 'δv', 'δz' be the Corresponding increments in 'u', 'v' and 'z' resp.

$$\Rightarrow \delta z = \delta x + i\delta y \qquad (\because z = x+iy).$$

$$f(z+\delta z) = (u+\delta u) + i(v+\delta v)$$

Now, $$f'(z) = \lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z\to 0} \frac{[(u+\delta u)+i(v+\delta v)] - [u+iv]}{\delta z}$$

$$f'(z) = \lim_{\delta z\to 0} \frac{\delta u + i\delta v}{\delta z} = \lim_{\delta z\to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right) \longrightarrow ②$$

$$\therefore f'(z) = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z}.$$

Since $w = f(z)$ is analytic in the region R, hence $f'(z)$ given by ② should have a unique value in $\delta z \longrightarrow 0$.

Let us Consider $\delta z \longrightarrow 0$ along a line Parallel to x-axis $\Rightarrow \delta y = 0$.

$$\therefore \delta z = \delta x \qquad (\because \delta z = \delta x + i\delta y)$$

$$\Rightarrow \delta x \to 0 \qquad (\because \delta z \to 0).$$

from ② $\Rightarrow$ $$f'(z) = \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \longrightarrow ③$$

Again let us consider $\delta z \to 0$ along a line parallel to y-axis $\Rightarrow \delta x = 0$.

$\therefore \delta z = i\delta y \qquad (\because \delta z = \delta x + i\delta y)$

$\Rightarrow i\delta y \to 0 \Rightarrow \delta y \to 0 \qquad (\because \delta z \to 0)$

from ② $\Rightarrow f'(z) = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\delta y} + i \frac{\delta v}{i\delta y} \right)$

$\Rightarrow f'(z) = \lim_{\delta y \to 0} \left(\frac{1}{i} \frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right)$

$\Rightarrow f'(z) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

$\Rightarrow f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \left(\because \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = -i \right) \longrightarrow ④$

from ③ & ④

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating real and imaginary parts we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow ⑤ \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow ⑦.$$

Hence, if $f(z)$ is analytic then the Cauchy - Riemann Eqns are satisfied.

Sufficient Condition:—

Conversely suppose that $f(z)$ is any function satisfying the given condition (i) & (ii).

We have to prove that $f(z)$ is analytic in a region 'R'.

$\Rightarrow f'(z)$ exists.

Now, $f(z+\delta z) = u(x+\delta x, y+\delta y) + i\, v(x+\delta x, y+\delta y)$

($\because$ Taylor's theorem for a fun of two var. is

~~f(x+y) = f(x,y)+~~

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y)$$

$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y) + \cdots\cdots \Big)$$

By applying Taylor's theorem and neglecting the second and higher order terms we have,

$$f(z+\delta z) = u(x,y) + \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y}\right) u(x,y) + i\left[v(x,y) + \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y}\right) v(x,y)\right]$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)\right]$$

$$f(z+\delta z) = u(x,y) + i v(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + i\left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)$$

$$f(z+\delta z) = f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$\Rightarrow f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\delta y \quad \left(\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y)$$

$$\Rightarrow f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z \qquad (\because \delta z = \delta x + i\delta y)$$

$$\Rightarrow \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

This shows that $f'(z)$ exists $\left(\because \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \text{ are exists}\right)$.

$\Rightarrow f(z)$ is analytic.

CAUCHY - RIEMANN EQUATIONS IN POLAR FORM:-

Let (r,θ) be the polar co-ordinates of the point (x,y).

$\therefore\ x = r\cos\theta\ ;\ y = r\sin\theta$

Since, $z = x + iy$

$\Rightarrow z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\Rightarrow z = e^{i\theta}.r$

Since, $\omega = f(z)$

$\Rightarrow u + iv = f(re^{i\theta}) \longrightarrow (1)$. $\quad (\because \omega = u + iv)$

Diff. (1) w.r. to 'r' Partially we have,

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta} \longrightarrow (2)$$

Again diff. (1) w.r.to 'θ' we have,

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}).$$

$$\frac{\partial}{\partial r}(u+iv) = \frac{\partial}{\partial r}\left[f(re^{i\theta})\right]$$

$$\Rightarrow \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot \frac{\partial}{\partial r}(re^{i\theta})$$

$$\Rightarrow \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \longrightarrow (2)$$

Again diff. (1) w.r.to 'θ' partially we have,

$$\frac{\partial}{\partial \theta}(u+iv) = \frac{\partial}{\partial \theta}\left[f(re^{i\theta})\right]$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot \frac{\partial}{\partial \theta}(re^{i\theta})$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot i \cdot re^{i\theta} \quad (3)$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \quad (\because \text{from } (2))$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts we have,

$$\boxed{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}} \quad \text{and} \quad \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

These are called C-R Equations in polar form.

It can also be written as.

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \& \quad \boxed{\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}}$$

NOTE:(2):- If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace Equations in 'D'.

i.e., $\nabla^2 u = 0$ & $\nabla^2 v = 0$.

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

(OR) S.T. both the real and imaginary Parts of an analytic function are Harmonic.

Harmonic function:- Any function $\phi(x,y)$ which possesses continuous Partial derivatives of the first and Second orders and satisfy Laplace equation $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ is called a "harmonic function".

It can also be written as $\nabla^2\phi=0$ where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called "Laplacian Operator".

Conjugate Harmonic functions:-

If a function $u(x,y)$ is harmonic in the domain and if we can find another harmonic function $v(x,y)$, such that they satisfy the Cauchy Riemann Equations and Laplace's Equations then we say that $v(x,y)$ is the "harmonic conjugate" of $u(x,y)$.

NOTE: ① If a harmonic function is given then the other harmonic conjugate function can be determined by using C-R Equations.

②

Problem:- Prove that the function $f(z)=\bar{z}$ is not analytic at any point.

Sol:- Let $\omega=f(z)=u+iv$ where $z=x+iy$.

Since, $z=x+iy \Rightarrow \bar{z}=x-iy$.

Given $f(z)=\bar{z}=x-iy$.

$\Rightarrow u=x \;;\; v=-y$

$\therefore \frac{\partial u}{\partial x}=1 \;;\; \frac{\partial v}{\partial x}=0$

$\frac{\partial u}{\partial y}=0 \;;\; \frac{\partial v}{\partial y}=-1.$

$\therefore \frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

C-R Eqns are not satisfied.

Hence $f(z)=\bar{z}$ is not analytic at any point.

Pro:- Determine whether the function $2xy + i(x^2-y^2)$, is analytic.

Sol:- Let $f(z) = u + iv$ where $z = x+iy$.

Given, $f(z) = 2xy + i(x^2-y^2)$

$$\Rightarrow u = 2xy \quad ; \quad v = x^2-y^2$$

$$\frac{\partial u}{\partial x} = 2y \quad ; \quad \frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 2x \quad ; \quad \frac{\partial v}{\partial y} = -2y$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C-R Equ's are not satisfied.

Hence $f(z) = 2xy + i(x^2-y^2)$ is not analytic.

Pro:- Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2$ where $\omega = f(z)$ is analytic.

Sol:- Let $f(z) = u + iv \longrightarrow (1)$ where $z = x+iy$.

Real part of $f(z) = u$.

Diff (1) w.r.to 'x' partially we have,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow |f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$\Rightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \longrightarrow (2)$$

We know $\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(u^2)\right] = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(u^2) = 2\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right] \longrightarrow (3)$$

Similarly we have, $\frac{\partial^2}{\partial y^2}(u^2) = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right] \longrightarrow (4)$

③ + ④ $\Rightarrow \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] \longrightarrow ⑤$

Since, 'u' is real part of $f(z)$ $\Rightarrow \nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. ($\because f(z)$ is analytic)

from ⑤ $\Rightarrow \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2) = 2|f'(z)|^2$ ($\because$ By ②)

$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Real } f(z)|^2 = 2|f'(z)|^2.$

Hence proved.

Pro:- If $f(z)$ is a regular function of z, prove that

(B.S.Grewal)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2. \quad \text{OR} \quad \nabla^2\left[|f(z)|^2\right] = 4|f'(z)|^2.$$

Sol:- Let $f(z) = u(x,y) + iv(x,y)$ be a regular function. where $z = x+iy$

$\Rightarrow f(z)$ is analytic everywhere in the whole complex plane.

Now, $|f(z)| = \sqrt{u^2+v^2}$

$\Rightarrow |f(z)|^2 = u^2+v^2 = \phi(x,y)$, say $\longrightarrow ①$

Now, $\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 2\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right]$

$\frac{\partial^2 \phi}{\partial x^2} = 2\left[\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial x} + v\cdot\frac{\partial^2 v}{\partial x^2}\right]$

$\therefore \frac{\partial^2 \phi}{\partial x^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2 + v\cdot\frac{\partial^2 v}{\partial x^2}\right] \longrightarrow ②$

III^{ly} we have, $\frac{\partial^2 \phi}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2 + v\cdot\frac{\partial^2 v}{\partial y^2}\right] \longrightarrow ③$

② + ③ $\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right]$

$+ 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \longrightarrow ④$

Since, $f(z)$ is analytic then 'u' and 'v' should satisfy C-R Eqns & also Laplace Equations.

i.e., $\nabla^2 u = 0$ & $\nabla^2 v = 0$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

from (4) $\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$

$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$ $\left[\because \begin{array}{l} z = u+iv \\ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ |f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} \end{array}\right]$

$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$. (∵ By (1)).

Hence proved//..

Pro:- Prove that the function z^n (n is a positive integer) is analytic and find its derivative...

Sol:- ~~Let us assume~~ Given, $f(z) = z^n$. where $z = x+iy$.

~~Let $f(z) = u(x,y) + i\,v(x,y)$~~

~~Converting it into polar co-ordinates we have,~~

~~$x = r\cos\theta$; $y = r\sin\theta$.~~

Given $f(z) = z^n$ where $z = x+iy$.

Let $f(z) = u(x,y) + i\,v(x,y)$

Let (r,θ) be the polar co-ordinates of (x,y).

$\therefore\ x = r\cos\theta$; $y = r\sin\theta$.

Hence, $f(z) = u(r,\theta) + i\,v(r,\theta)$. $\rightarrow$ (1)

$z = x+iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

$z = re^{i\theta}$.

$\therefore\ f(z) = z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta) = r^n\cos n\theta + i\,r^n\sin n\theta$ (2)

from (1) & (2) $\quad u(r,\theta) = r^n\cos n\theta$; $v(r,\theta) = r^n\sin n\theta$.

$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta$; $\frac{\partial v}{\partial r} = n\cdot r^{n-1}\sin n\theta$

$\frac{\partial u}{\partial \theta} = r^n(-n\sin n\theta)$; $\frac{\partial v}{\partial \theta} = r^n(n\cos n\theta)$.

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\cdot n r^n \cos n\theta = n r^{n-1}\cos n\theta = \frac{\partial u}{\partial r}.$$

$$\therefore \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

Also, $\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{1}{r}\cdot(-n\cdot r^n \sin n\theta) = -n r^{n-1}\sin n\theta = -\frac{\partial v}{\partial r}.$

$$\therefore \boxed{\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}}$$

C-R Equis. are satisfied.

$\Rightarrow f(z) = z^n$ is analytic.

We have, $\frac{\partial f}{\partial r} = \frac{df}{dz}\cdot\frac{\partial z}{\partial r} = f'(z)\, e^{i\theta}$ $\quad \left(\because z = re^{i\theta} \Rightarrow \frac{\partial z}{\partial r} = e^{i\theta}\right)$

$$\Rightarrow f'(z) = \frac{1}{e^{i\theta}}\cdot\frac{\partial f}{\partial r} = e^{-i\theta}\frac{\partial f}{\partial r} = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\Rightarrow f'(z) = e^{-i\theta}\left(n r^{n-1}\cos n\theta + i\, n r^{n-1}\sin n\theta\right)$$

$$\Rightarrow f'(z) = e^{-i\theta}\cdot n r^{n-1}(\cos n\theta + i\sin n\theta) = e^{-i\theta}\cdot n r^{n-1}\cdot e^{in\theta}$$

$$\Rightarrow f'(z) = n r^{n-1} e^{i(n-1)\theta} = n\left(r e^{i\theta}\right)^{n-1} = n z^{n-1}. \quad (\because z = re^{i\theta}).$$

$$\therefore f'(z) = n z^{n-1}$$

Pro:- Determine 'p' such that the function $f(z) = \frac{1}{2}\log(x^2+y^2) + i\,\mathrm{Tan}^{-1}\left(\frac{px}{y}\right)$ be an analytic function.

Sol:- $P = -1$ (Apply C-R Equis. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$).

4) Proove that the function $f(z)$ defined by $f(z) =$

$$\begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & (z \neq 0) \\ 0, & (z=0) \end{cases}$$

is continuous and the cauchy's Riemann eq^n are satisfied at the origin, yet $y'(0)$ doesnot exist.

Sol^n: Given $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z=0 \end{cases}$

i) To proove that $f(z)$ is continuous

We know that the definition of continuity of a function f is continuous at $z=a$

if $\underset{z \to a}{Lt} f(z) = f(a)$

for $z=0$, if $\underset{z \to 0}{Lt} f(z) = f(0)$

$$\underset{z \to 0}{Lt} f(z) = \underset{z \to 0}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$= \underset{\substack{x \to 0 \\ y \to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

Since the limit exist ~~it~~ must have the same value independence of path along which $z \to 0$.

We consider the following paths i) along x-axis

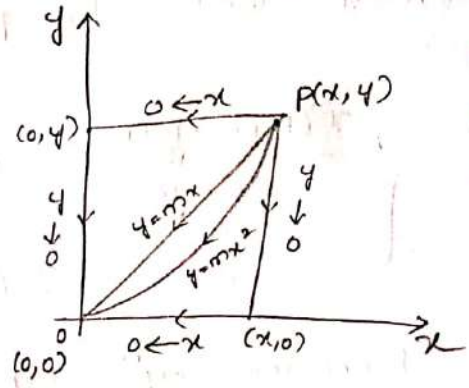
ii) along y-axis

iii) Along the line $y = mx$

iv) Along the path $y = mx^2$

[i) To prove that $f(z)$ i:]

i) Along the x-axis:-

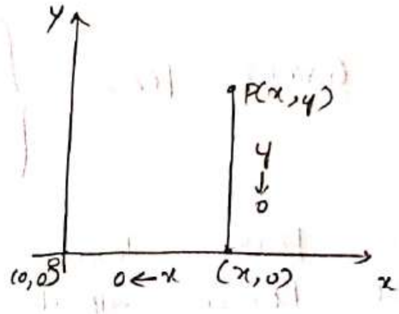


Let $y \to 0$ first and then $x \to 0$

$$\underset{z\to 0}{Lt}\, f(z) = \underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \underset{x\to 0}{Lt} \frac{x^{\cancel{3}}(1+i)}{\cancel{x^2}}$$

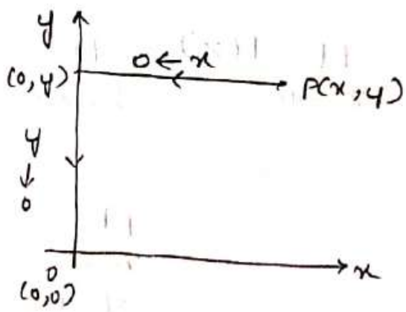
$$= \underset{x\to 0}{Lt}\, x(1+i) = 0$$



$\therefore \underset{z\to 0}{Lt}\, f(z) = 0 = f(0)$ [$\because$ By the def^n of $f(z)$]

ii) Along the y-axis:-

Let $x \to 0$ first and then $y \to 0$



$$\underset{z\to 0}{Lt}\, f(z) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

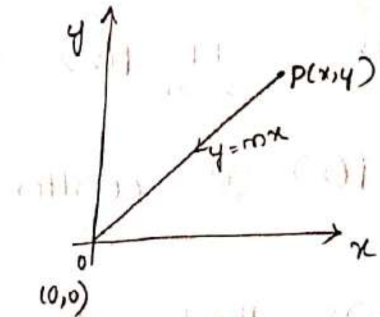
$$= \underset{y\to 0}{Lt} \frac{-y^{\cancel{3}}(1-i)}{\cancel{y^2}}$$

$$= \underset{y\to 0}{Lt} -y(1-i) = 0$$

$\therefore \underset{z\to 0}{Lt}\, f(z) = 0 = f(0)$ [$\because$ By the def^n of $f(z)$]

iii) Along the line $y=mx$:-

Let $y=mx$ first and then $x\to 0$



$$\underset{z\to 0}{Lt}\, f(z) = \underset{z\to 0}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$= \underset{\substack{y=mx\\ x\to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$= \underset{x\to 0}{Lt} \frac{x^3(1+i)-m^3x^3(1-i)}{x^2+m^2x^2}$$

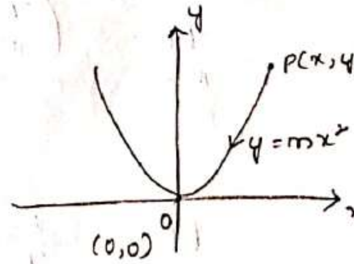
$$= \underset{x\to 0}{Lt} \frac{x^{\cancel{3}}((1+i)-m^3(1-i))}{\cancel{x^2}(1+m^2)}$$

$$= \underset{x\to 0}{Lt} \frac{x((1+i)-m^3(1-i))}{(1+m^2)} = 0$$

$\therefore \underset{z\to 0}{Lt}\, f(z) = 0 = f(0)$ [$\because$ By the def^n of $f(z)$]

iv) Along the line $y=mx^2$

Let $y=mx^2$ first & then $x\to 0$



$$\underset{z\to 0}{Lt}\, f(z) = \underset{z\to 0}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$= \underset{\substack{y=mx^2\\ x\to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$= \underset{x\to 0}{Lt} \frac{x^3(1+i)-m^3x^6(1-i)}{x^2+m^2x^4}$$

$$= \underset{x\to 0}{Lt} \frac{x^{\cancel{3}}[(1+i)-m^3x^3(1-i)]}{\cancel{x^2}(1+m^2x^2)} = 0$$

$\therefore \underset{z\to 0}{Lt} f(z) = 0 = f(0)$ [$\because$ By def^n of $f(z)$]

We observe that from above four cases, along all the paths $\underset{z\to 0}{Lt} f(z)$ value is same i.e, $\underset{z\to 0}{Lt} f(z) = f(0)$

$\therefore f(z)$ is continuous at $z=0$

2) To prove that $f(z)$ satisfies C-R eq^{n}s at the origin:-

Given $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$

$$= \left(\frac{x^3-y^3}{x^2+y^2}\right) + i\left(\frac{x^3+y^3}{x^2+y^2}\right), \ z\neq 0$$

Let $f(z) = u(x,y) + iv(x,y)$

so $u(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$, $v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

$$\left(\frac{\partial u}{\partial x}\right)_{at\ (0,0)} = \underset{x\to 0}{Lt} \frac{u(x,0) - u(0,0)}{x}$$

$$= \underset{x\to 0}{Lt} \frac{\left(\frac{x^3}{x^2}\right) - 0}{x}$$

$$= \underset{x\to 0}{Lt} \frac{\cancel{x}}{\cancel{x}} = 1$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{at\ (0,0)} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{at\ (0,0)} = \underset{y\to 0}{Lt} \frac{u(0,y) - u(0,0)}{y}$$

$$= \underset{y\to 0}{Lt} \frac{-y-0}{y}$$

$$= -1$$

$$\therefore \left(\frac{\partial u}{\partial y}\right)_{(0,0)} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{at\ (0,0)} = \underset{x\to 0}{Lt} \frac{v(x,0)-v(0,0)}{x}$$

$$= \underset{x\to 0}{Lt} \frac{x-0}{x}$$

$$= \underset{x\to 0}{Lt} \frac{x}{x}$$

$$= 1$$

$$\therefore \left(\frac{\partial v}{\partial x}\right)_{at\ (0,0)} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{at\ (0,0)} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y}$$

$$= \underset{y\to 0}{Lt} \frac{y-0}{y}$$

$$= \underset{y\to 0}{Lt} \frac{y}{y} = 1$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_{at\ (0,0)} = 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$\therefore$ C-R eq^{n}s are satisfied at the origin.

3) To prove that $f'(0)$ doesnot exist

We know that the derivation of the function f at $z=a$ is $f'(a) = \underset{z\to a}{Lt} \dfrac{f(z)-f(a)}{z-a}$

At $z=0$, $f'(0) = \underset{z\to 0}{Lt} \dfrac{f(z)-f(0)}{z}$

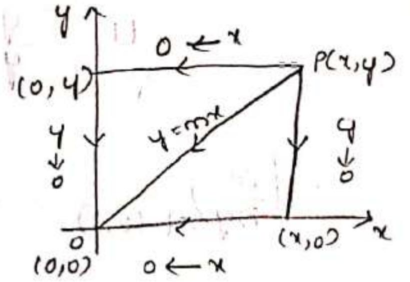
$$= \underset{z\to 0}{Lt} \frac{\left[\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} - 0\right]}{z}$$

$$= \underset{z\to 0}{Lt} \frac{\left(\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}\right)}{z}$$

$$f'(0) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)z}$$

$$f'(0) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)}$$

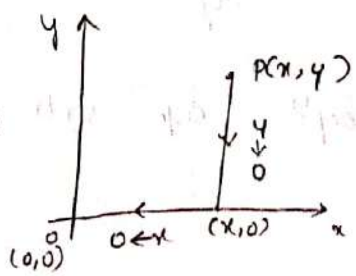
Since the limit exists, it must have the same limit value independent of the path along which $z\to 0$, we consider the following paths.



i) Along x-axis
ii) Along y-axis
iii) Along the line $y=mx$
[iv) Along]

i) Along the x-axis :-

Let $y\to 0$ first then $x\to 0$



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$\therefore f'(0) = \underset{\substack{x\to 0\\ y\to 0\\ z\to 0}}{Lt} \dfrac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$

$= \underset{\substack{y\to 0\\ x\to 0}}{Lt} \dfrac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$

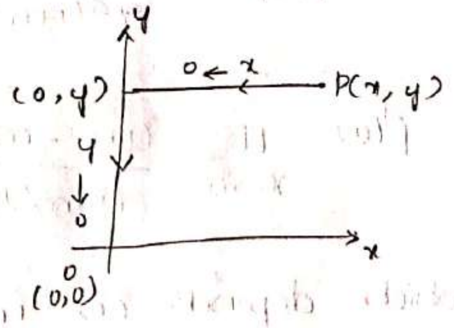
$= \underset{x\to 0}{Lt} \dfrac{x^3(1+i)}{x^2(x)}$

$= \left[\underset{x\to 0}{Lt_x}\right] = 1+i$

$\therefore f'(0) = 1+i$

ii) Along the y-axis :-

Let $x\to 0$ first and then $y\to 0$



$\therefore f'(0) = \underset{z\to 0}{Lt} \dfrac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$

$= \underset{\substack{x\to 0\\ y\to 0}}{Lt} \dfrac{x^3(1+i) - y^3(1-i)}{(x^2+y^2)(x+iy)}$

$= \underset{y\to 0}{Lt} \dfrac{-y^3(1-i)}{y^3 i}$

$= \underset{y\to 0}{Lt} \dfrac{i-1}{i} = \dfrac{i}{i} - \dfrac{1}{i}$

$= 1+i$

$f'(0) = 1+i$

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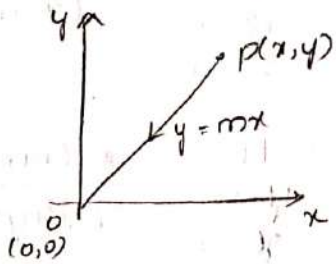
iii) Along the line $y=mx$:-

Let $y=mx$ first and $x\to 0$



$$f'(0) = \underset{z\to 0}{Lt} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)}$$

$$= \underset{\substack{y=mx \\ x\to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)}$$

$$= \underset{x\to 0}{Lt} \frac{x^3(1+i)-m^3x^3(1-i)}{(x^2+m^2x^2)(x+imx)}$$

$$= \underset{x\to 0}{Lt} \frac{\cancel{x^3}\left[(1+i)-m^3(1-i)\right]}{\cancel{x^3}(1+m^2)(1+im)}$$

$$\therefore f'(0) = \underset{x\to 0}{Lt} \frac{(1+i)-m^3(1-i)}{(1+m^2)(1+im)}$$

which depends on m and hence is not unique.

$\therefore$ $f'(z)$ doesnot exist at origin i.e., $f'(0)$ doesnot exist.

5) show that the function $f(z)=\sqrt{|xy|}$ is not analytic at the origin, although cauchy's - Riemann eq^n are satisfied at that point.

sol^n: Given $f(z)=\sqrt{|xy|}$

i) $f(z)$ is not analytic (not differentiable)

we know that the derivation of the function f at $z=a$

is $f'(a) = \underset{z\to a}{Lt} \frac{f(z)-f(a)}{z-a}$

At $z=0$, $f'(0) = \underset{z\to 0}{Lt} \dfrac{f(z)-f(0)}{z}$

$$= \underset{z\to 0}{Lt} \frac{\sqrt{|xy|}-0}{z}$$

$$f'(0) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{\sqrt{|xy|}}{(x+iy)}$$

since the limit exists, it must have the same limit value independent of the path along which $z\to 0$, we consider the following paths.

i) Along x-axis
ii) Along y-axis
iii) Along the line $y=mx$

i) Along the x-axis:-

Let $y\to 0$ first then $x\to 0$

$$f'(0) = \underset{z\to 0}{Lt} \frac{\sqrt{|xy|}}{x+iy}$$

$$= \underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{\sqrt{|xy|}}{x+iy}$$

$$f'(0) = 0$$

ii) Along the y-axis:-

Let $x\to 0$ first then $y\to 0$

$$f'(0) = \underset{z\to 0}{Lt} \frac{\sqrt{|xy|}}{x+iy}$$

$$= \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{\sqrt{|xy|}}{x+iy}$$

$f'(0) = 0$

iii) Along the line $y = mx$:-

Let $y = mx$ first and $x \to 0$

$$f'(0) = \underset{z \to 0}{Lt} \frac{\sqrt{|xy|}}{x+iy}$$

$$= \underset{\substack{y=mx \\ x \to 0}}{Lt} \frac{\sqrt{|xy|}}{x+iy}$$

$$= \underset{x \to 0}{Lt} \frac{\sqrt{|x \cdot mx|}}{x+imx}$$

$$= \underset{x \to 0}{Lt} \frac{\cancel{x}\sqrt{|m|}}{\cancel{x}(1+im)}$$

$$f'(0) = \underset{x \to 0}{Lt} \frac{\sqrt{|m|}}{(1+im)}$$

which depends on m and hence it is not unique.

$\therefore$ $f'(z)$ doesnot exist at origin i.e., $f'(0)$ doesnot exist.

2) To prove that $f(z)$ satisfies C-R eq^{n}s at the origin :-

Given $f(z) = \sqrt{|xy|}$

$$= \sqrt{|xy|} + i(0)$$

Let $f(z) = u(x,y) + iv(x,y)$

so $u(x,y) = \sqrt{|xy|}$, $v(x,y) = 0$

$$\left(\frac{\partial u}{\partial x}\right)_{at\ (0,0)} = \underset{x \to 0}{Lt} \frac{u(x,0) - u(0,0)}{x}$$

$$= \underset{x \to 0}{Lt} \frac{0-0}{x} = 0$$

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$$\therefore \left(\frac{\partial u}{\partial x}\right)_{at\ (0,0)} = 0$$

$$\left(\frac{\partial u}{\partial y}\right)_{at\ (0,0)} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{0-0}{y} = 0$$

$$\therefore \left(\frac{\partial u}{\partial y}\right)_{at\ (0,0)} = 0$$

$$\left(\frac{\partial v}{\partial x}\right)_{at\ (0,0)} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$\therefore \left(\frac{\partial v}{\partial x}\right)_{at\ (0,0)} = 0$$

$$\left(\frac{\partial v}{\partial y}\right)_{at\ (0,0)} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= \lim_{y \to 0} \frac{0-0}{y} = 0$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_{at\ (0,0)} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ C-R eq^{n}s are satisfied at the origin.

7) If $f(z)$ is analytic function then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Sol^n: Let $z = x+iy$

then $\bar{z} = x-iy$

Now $z+\bar{z} = x+\cancel{iy}+x-\cancel{iy}$

$= 2x$

$2x = z+\bar{z}$

$$\boxed{x = \frac{z+\bar{z}}{2}}$$

and $z-\bar{z} = \cancel{x}+iy-\cancel{x}+iy$

$z-\bar{z} = 2iy$

$y = \frac{1}{2i}(z-\bar{z})$

$$\boxed{y = \frac{-i}{2}(z-\bar{z})}$$

Here $x = \frac{1}{2}(z+\bar{z})$, $y = -\frac{i}{2}(z-\bar{z})$

$\frac{\partial x}{\partial z} = \frac{1}{2}$ $\qquad$ $\frac{\partial y}{\partial z} = \frac{-i}{2}$

$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$ $\qquad$ $\frac{\partial y}{\partial \bar{z}} = \frac{i}{2}$

We know that $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}$

$= \frac{\partial f}{\partial x}\cdot\frac{1}{2}+\frac{\partial f}{\partial y}\cdot\left(\frac{-i}{2}\right)$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

similarly, $\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$

$$= \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\cdot\left(\frac{i}{2}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$

$$\therefore \frac{\partial^2 f}{\partial z\partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right)$$

$$= \frac{\partial}{\partial z}\left(\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f$$

$$\therefore \frac{\partial^2}{\partial z\partial \bar{z}}f = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f$$

$$\frac{\partial^2}{\partial z\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\left(\frac{\partial^2}{\partial z\partial \bar{z}}\right) \quad \text{—} (1)$$

Now $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\cdot\frac{\partial^2}{\partial z\partial \bar{z}}|f(z)|^2$

$$= 4\cdot\frac{\partial^2}{\partial z\partial \bar{z}}\left[f(z)\cdot\overline{f(z)}\right] \quad \left[\because |f(z)|^2 = f(z)\cdot\overline{f(z)}\right]$$

$$= 4\cdot\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\overline{f(z)}\right]f(z)$$

$$= 4 \cdot \frac{\partial}{\partial z}\left[\overline{f'(z)}\right] f(z)$$

$$= 4 \cdot \overline{f'(z)} \left[\frac{\partial}{\partial z} f(z)\right]$$

$$= 4 \left(\overline{f'(z)} \; f'(z)\right) = 4 |f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4 |f'(z)|^2.$$

Harmonic functions - Laplace equation:-

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain 'D', then u and v satisfy Laplace eqn.

$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and

$\nabla^2 V = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ respectively in 'D' and have continuous second order partial derivatives in 'D'.

Harmonic function:-

Solutions of Laplace eqns having continuous second order partial derivatives are called Harmonic functions. There theory is called Potential theory hence, the real and imaginary parts of an analytic function are Harmonic functions.

Thus the function satisfying Laplace eqn $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ are known as Harmonic functions.

Conjugate Harmonic function :-

If two harmonic functions u and v satisfy the cauchy's Riemann eqn in domain 'D' and they are the real and imaginary parts of an analytic function 'f' in D. Then v is said to be a conjugate harmonic function of u in D.

Two harmonic functions, u and v which are such that u+iv is an analytic function are called conjugate harmonic functions.

In other words $f(z) = u+iv$ is analytic and if u and v satisfy laplace eqn, then u and v are called conjugate harmonic functions.

1) Prove that if $u = x^2-y^2$, $v = \frac{-y}{x^2+y^2}$ both u and v satisfy Laplace's eqn, but u+iv is not a regular function (analytic function of z].

Soln: Given $u = x^2-y^2$, $v = \frac{-y}{x^2+y^2}$

Now we have to prove that u, v satisfies the Laplace eqn i.e, $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial^2 u}{\partial x^2} = 2$

$\frac{\partial u}{\partial y} = -2y$, $\frac{\partial^2 u}{\partial y^2} = -2$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2 = 0$

$\therefore$ u satisfies Laplace equation.

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(0) + y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2xy}{(x^2+y^2)^2}\right)$$

$$= \frac{(x^2+y^2)^2(2y) - 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$$\boxed{\frac{\partial^2 v}{\partial x^2} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

and $$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

$$= \frac{(x^2+y^2)^2(2y) - (y^2-x^2) \cdot 2(x^2+y^2) \cdot (2y)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 4y(y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{2x^2y+2y^3-4y^3+4x^2y}{(x^2+y^2)^3} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}$$

$$= \frac{-(2y^3-6x^2y)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = \frac{2y^3-6x^2y}{(x^2+y^2)^3} - \frac{(2y^3-6x^2y)}{(x^2+y^2)^3} = 0$$

$\therefore$ v satisfies the laplace eqn.

Hence both u and v satisfies the laplace eqn.

We observe that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

Since u and v donot satisfy the C-R eqns,

$\therefore$ u+iv is not an analytic function (Regular function of z)

2) Show that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

3) Show that $u = x^2-y^2-y$ is harmonic.

Milne - Thomson method :-

construction of analytic function whose real or imaginary part are known.

Suppose $f(z) = u+iv$ is an analytic function, whose real part u is known. we can find v, the imaginary part and also the function f(z).

Consider $f(z) = u+iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Here $f(z)$ is an analytic function, so it satisfies C R eq^{ns}

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\therefore f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$f'(z) = \phi_1(x,y) + i\,\phi_2(x,y)$

Now $f'(z)$ is expressed in terms of z by replacing x by z and y by zero.

$\therefore f'(z) = \phi_1(z,0) + i\,\phi_2(z,0)$

Now integrating, we get

$$f(z) = \int(\phi_1(z,0) + i\,\phi_2(z,0))dz + c$$

where c is a complex constant.

Problems :-

i) Find the analytic function whose real part is

i) $\frac{x}{x^2+y^2}$

ii) $\frac{y}{x^2+y^2}$

Sol^n: i) Given $u = \frac{x}{x^2+y^2}$

Let $f(z) = u+iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ $\left[\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$ —— (1)

$$u = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

and $$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

Now sub the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ in eq^n (1)

$$(1) \Rightarrow f'(z) = \frac{y^2-x^2}{(x^2+y^2)^2} - i\frac{(-2xy)}{(x^2+y^2)^2}$$

$$f'(z) = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{i(2xy)}{(x^2+y^2)^2}$$

By Milne-Thomson method express $f'(z)$ in terms of z, replace x by z and y by zero.

$$f'(z) = \frac{-\cancel{z^2}}{(z^2)^{\cancel{2}}} + i(0)$$

$$f'(z) = \frac{-1}{z^2}$$

Now integrating, we get

$$f(z) = \int -\frac{1}{z^2}\,dz + c$$

$$= -\left[-\frac{1}{z}\right] + c$$

$f(z) = \frac{1}{z} + c$, where c is a complex constant.

Note:- This is the required analytic function.

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

Q) Show that $u(x,y) = e^{2x}(x\cos 2y - y\sin 2y)$ is harmonic and find its harmonic conjugate.

(or)

Find the analytic function whose real part is $u(x,y) = e^{2x}(x\cos 2y - y\sin 2y)$.

Given, $u(x,y) = e^{2x}(x\cos 2y - y\sin 2y) = xe^{2x}\cos 2y - e^{2x}y\sin 2y$

$$\frac{\partial u}{\partial x} = \cos 2y\left[xe^{2x}(2) + e^{2x}\right] - y\sin 2y \cdot 2e^{2x}$$

$$\frac{\partial u}{\partial x} = 2xe^{2x}\cos 2y + e^{2x}\cos 2y - 2ye^{2x}\sin 2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2\cos 2y\left[xe^{2x}(2) + e^{2x}\right] + \cos 2y(e^{2x}\cdot 2] - 2y\sin 2y(e^{2x}\cdot 2)$$

$$= 4xe^{2x}\cos 2y + 2\cos 2y e^{2x} + 2e^{2x}\cos 2y - 4ye^{2x}\sin 2y$$

$$= 4xe^{2x}\cos 2y + 4e^{2x}\cos 2y - 4ye^{2x}\sin 2y$$

$$\frac{\partial^2 u}{\partial x^2} = 4e^{2x}(x+1)\cos 2y - 4ye^{2x}\sin 2y$$

$$\frac{\partial u}{\partial y} = xe^{2x}(-\sin 2y \cdot 2) - e^{2x}\left[y \cdot 2\cos 2y + \sin 2y\right]$$

$$\frac{\partial u}{\partial y} = -2xe^{2x}\sin 2y - 2ye^{2x}\cos 2y - e^{2x}\sin 2y$$

$$\frac{\partial^2 u}{\partial y^2} = -2xe^{2x}(\cos 2y \cdot 2) - 2e^{2x}\left[y(-2\sin 2y) + \cos 2y\right] - e^{2x}(2\cos 2y)$$

$$\frac{\partial^2 u}{\partial y^2} = -4xe^{2x}\cos 2y + 4ye^{2x}\sin 2y - 2e^{2x}\cos 2y - 2e^{2x}\cos 2y$$

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$$\frac{\partial^2 u}{\partial y^2} = -4xe^{2x}\cos 2y - 4e^{2x}\cos 2y + 4ye^{2x}\sin 2y$$

$$\frac{\partial^2 u}{\partial y^2} = -4e^{2x}(x+1)\cos 2y + 4ye^{2x}\sin 2y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is a Harmonic function

let $f(z) = u + iv$ be an analytic function

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = \left(e^{2x}(1+2x)\cos 2y - 2ye^{2x}\sin 2y\right) - i\left(-e^{2x}(1+2x)\sin 2y - 2ye^{2x}\cos 2y\right)$$

By Milne Thomson method express $f'(z)$ completely in terms of z, replace x by z and y by zero

$$f'(z) = \left(e^{2z}(1+2z)\cos(0) - 0\right) - i\left(-e^{2z}(1+2z)\sin 0 - 0\right)$$

$$f'(z) = e^{2z}(1+2z) - i(0)$$

$$f'(z) = (1+2z)e^{2z} = e^{2z} + 2ze^{2z}$$

Now integrating we get

$$f(z) = \int e^{2z}\,dz + 2\int ze^{2z}\,dz + c$$

$$= \frac{e^{2z}}{2} + 2\left[z \cdot \frac{e^{2z}}{2} - \int \frac{e^{2z}}{2}\,dz\right] + c$$

$$= \frac{e^{2z}}{2} + 2\left[\frac{ze^{2z}}{2} - \frac{1}{4}e^{2z}\right] + c$$

$= \frac{1}{2}\cancel{e^{2z}} + ze^{2z} - \frac{1}{2}\cancel{e^{2z}} + c$

$\therefore \; f(z) = ze^{2z} + c$

This is the required analytic function.

Let $z = x+iy$

$f(z) = (x+iy)e^{2(x+iy)}$

$= (x+iy) \cdot e^{2x} \cdot e^{i2y}$

$= e^{2x}(x+iy)(\cos 2y + i\sin 2y)$

$= e^{2x}(x\cos 2y + ix\sin 2y + iy\cos 2y - y\sin 2y) + c$

$= e^{2x}(x\cos 2y - y\sin 2y) + ie^{2x}(x\sin 2y + y\cos 2y) + c$

i.e., $f(z) = u+iv$

$v(x,y) = e^{2x}(x\sin 2y + y\cos 2y)$ is conjugate harmonic of $u(x,y)$

3) Show that the function $u = 4xy - 3x + 2$ is harmonic. Construct the corresponding analytic function $f(z) = u+iv$ in terms of z.

Sol^n: Given $u = 4xy - 3x + 2$

$\frac{\partial u}{\partial x} = 4y - 3 \quad , \quad \frac{\partial u}{\partial y} = 4x$

$\frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is harmonic function.

Let $f(z) = u+iv$ be an analytic function

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = 4y-3 - i(4x)$$

By Milne Thomson method express $f'(z)$ completely in terms of z, replace x by z and y by zero.

$$f'(z) = 4(0)-3-i(4z)$$

$$f'(z) = -3 - i4z$$

Now integrating we get

$$f(z) = \int -3dz - i\int 4z dz + C$$

$$= -3z - i4\cdot\frac{z^2}{2} + C$$

$$f(z) = -3z - i2z^2 + C$$

$\therefore$ This is the required analytic function.

4) Show that the function $f(x,y) = x^3y - xy^3 + xy + x + y$ can be imaginary part of an analytic function of $\phi(z) = u+iv$. Determine the real part & also the complex function.

5) If $f(z) = u+iv$ is an analytic function of z and if $u-v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z

Sol^n: Given $u-v = e^x(\cos y - \sin y)$

let $f(z) = u+iv$ —— ①

$if(z) = iu - v$ —— ②

From eq^{n}s ① and ② we get

$f(z) + if(z) = u + iv + iu - v$

$(1+i)f(z) = (u-v) + i(u+v)$

$(1+i)f(z) = U + iV$ where $U = u - v$ & $V = u + v$.

Differentiating we get $(1+i)f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$ $\left[\because \text{C-R eq}^n\right]$ $\left[\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}\right]$

$= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) \mp i\left(\frac{\partial u}{\partial y} \mp \frac{\partial v}{\partial y}\right)$

$\left[= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)\right.$

$= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$ $\left[\because \text{By C-R eq}^n \; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]\Big]$

$\therefore (1+i)f'(z) = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)$ — ③

Now $u - v = e^x(\cos y - \sin y)$

$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y)$

$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x(-\sin y - \cos y)$

Now sub the above values in eq^n ③, we get

Eq^n ③ $\Rightarrow (1+i)f'(z) = e^x(\cos y - \sin y) - ie^x(-\sin y - \cos y)$

By Millen thomson method, expressing $(1+i)f'(z)$ completely in terms of z, replace x by z and y by 0.

$(1+i)f'(z) = e^z(\cos(0) - \sin(0)) - ie^z(-\sin(0) - \cos(0))$

$= e^z(1) - ie^z(-1)$

$= e^z + ie^z$

$(1+i)f'(z) = (1+i)e^z$

$f'(z) = e^z$

Now, integrating, we get

$$f(z) = \int e^z dz + c$$

$$f(z) = e^z + c$$

which is the required analytic function.

6) If $f(z) = u+iv$ is an analytic function of z and if $u-v = (x-y)(x^2+4xy+y^2)$, find $f(z)$ in terms of z.

Sol^n: Given $u-v = (x-y)(x^2+4xy+y^2)$

Let $f(z) = u+iv$ —— ①

$if(z) = iu - v$ —— ②

From eq^{n}s ① and ② we get

$f(z) + if(z) = u+iv+iu-v$

$(1+i)f(z) = u-v+i(u+v)$

$(1+i)f(z) = U+iV$ where $U = u-v$, $V = u+v$

differentiating we get $(1+i)f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$

$= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)$ $\left[\because \text{C-R eq}^n, \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}\right]$

$\therefore (1+i)f'(z) = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)$ —— ③

Now $u-v = (x-y)(x^2+4xy+y^2)$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = (x-y)(2x+4y) + (x^2+4xy+y^2)(1)$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = (x-y)(4x+2y) + (x^2+4xy+y^2)(-1)$$

Now sub the values in eq^n ③, we get

(3) $\Rightarrow$ $(1+i)f'(z) = (x-y)(2x+4y) + (x^2+4xy+y^2) - i\left[(x-y)(4x+2y) - (x^2+4xy+y^2)\right]$

$$= (2x^2+4xy-2xy-4y^2+x^2+4xy+y^2) - i[4x^2+2xy-4xy-2y^2 - x^2-4xy-y^2]$$

$$(1+i)f'(z) = (3x^2+6xy-3y^2) - i[3x^2-6xy-3y^2]$$

By Milne thomson method, expressing $(1+i)f'(z)$ completely in terms of z, replace x by z and y by 0.

$$(1+i)f'(z) = (3z^2+0) - i[3z^2]$$

$$= 3z^2 - i3z^2$$

$$(1+i)f'(z) = 3z^2(1-i)$$

$$f'(z) = 3z^2\frac{(1-i)}{(1+i)} \times \frac{(1-i)}{(1-i)}$$

$$= \frac{3z^2(\cancel{1}-\cancel{1}-2i)}{1^2+1}$$

$$= \frac{3z^2(-\cancel{2}i)}{\cancel{2}}$$

$$f'(z) = -i\,3z^2.$$

Now integrating, we get

$$f(z) = -i\int 3z^2 + c$$

$$f(z) = -i\cdot\frac{\cancel{3}z^3}{\cancel{3}} + c$$

$$f(z) = -iz^3 + c$$

Which is the required analytic function.

7) Find k, such that $f(x,y) = x^3 + 3kxy^2$ may be harmonic and find its conjugate

Soln: Given $f(x,y) = x^3 + 3kxy^2$

$$\frac{\partial f}{\partial x} = 3x^2 + 3ky^2$$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial f}{\partial y} = 6kxy$$

$$\frac{\partial^2 f}{\partial y^2} = 6kx$$

Given $f(x,y)$ is harmonic function.

i.e, $\nabla^2 f = 0$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$6x + 6kx = 0$$

$$6x(1+k) = 0$$

$$6x \neq 0, \quad 1+k = 0$$

$$\therefore k = -1$$

If $k = -1$, then $f(x,y)$ is harmonic

To find conjugate of f:-

Let $f(x,y) = u(x,y) = x^3 + 3kxy^2$

$$= x^3 + 3(-1)xy^2$$

$$= x^3 - 3xy^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy$$

Let $f(z) = u+iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$ $\left[\because \text{By C-R eq}^n \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$

$= (3x^2-3y^2) + i(-(-6xy))$

$f'(z) = (3x^2-3y^2) + i\,6xy$

By Milne-thompson method express $f'(z)$ completely in terms of z, replace x by z and y by 0.

$f'(z) = (3z^2 - 3(0)) + i(6z(0))$

$f'(z) = 3z^2$

Now integrating we get $f(z) = \int 3z^2 dz = \frac{\cancel{3}z^3}{\cancel{3}} + c$

$f(z) = z^3 + c$

Put $z = x+iy$

$f(z) = (x+iy)^3 + c$

$= x^3 + i^3y^3 + 3xi^2y^2 + 3x^2iy + c$

$= x^3 - iy^3 - 3xy^2 + i(3x^2y) + c$

$= (x^3 - 3xy^2) + i(3x^2y - y^3) + c$

$\therefore$ The conjugate of $f(x,y) = x^3 - 3x^2y$ is $3x^2y - y^3 + c$

8) show that $u(x,y) = x^3 - 3xy^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.

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sol^n: Given $f(x,y) = x^3 - 3xy^2$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial f}{\partial y} = -6xy$$

$$\frac{\partial^2 f}{\partial y^2} = -6x$$

To show that $f(x,y)$ is harmonic, we have to prove

$\nabla^2 f = 0 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

$\Rightarrow \quad 6x - 6x = 0$

$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

$\therefore f(x,y)$ is harmonic.

To find conjugate of f :-

Let $f(x,y) = u(x,y) = x^3 - 3xy^2$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = -6xy$

Let $f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right)$ [$\because$ By C-R eq^{n}s $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$]

$= (3x^2 - 3y^2) + i(-(-6xy))$

$f'(z) = (3x^2 - 3y^2) + i(6xy)$

By Milnethomson method express $f'(z)$ completely interms of 'z', Replace x by z and y by 0

$\therefore f'(z) = (3z^2 - 0) + i(0)$

$f'(z) = 3z^2$

Now integrating we get $f(z) = \int f'(z) dz$

$f(z) = \int 3z^2 dz$

$= \frac{\cancel{3} z^3}{\cancel{3}} + c$

$f(z) = z^3 + c$

Put $z = x + iy$

$f(z) = (x+iy)^3 + c$

$= x^3 + (iy)^3 + 3x^2 iy + 3x(iy)^2 + c$

$= x^3 - iy^3 + 3x^2 iy - 3xy^2 + c$

$= (x^3 - 3xy^2) + i(3x^2y - y^3) + c$

$\therefore$ The conjugate of $f(x,y) = x^3 - 3x^2y$ is $3x^2y - y^3 + c$

9) show that the function $u = a \log(x^2+y^2)$ is harmonic and find its conjugate.

10) show that both the real and imaginary parts of an analytic function are harmonic

10 sol^n: Let $f(z) = u + iv$ be an analytic function.

$\therefore$ By C-R eq^{n}s

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—(1)} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—(2)}$$

Now we have to prove that u, v are harmonic i.e, to prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Now differentiating eq^n (1) with respect x partially, we get.

$$(1) \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{—(3)}$$

Differentiating eq^n (2) partially with respect y,

$$\frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 v}{\partial x \partial y} \quad \text{—(4)}$$

From eq^{n}s (3) & (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is harmonic.

Now differentiating eq^n (1) partially with respect y.

$① \Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$ —— ⑤

Differentiating eq^n ③ partially with respect x

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{-\partial^2 v}{\partial x^2} \Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{-\partial^2 u}{\partial x \partial y} \text{ —— ⑥}$$

From eq^n ⑤ & ⑥ , we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{-\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$\therefore$ v is Harmonic.

Hence both the real and imaginary parts of an analytic function are Harmonic.

11) If $u(x,y)$ and $v(x,y)$ are harmonic functions in a region 'R', prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function.

Sol^n: Given u and v are harmonic functions, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{-\partial^2 u}{\partial x^2} \text{ —— ① and}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{-\partial^2 v}{\partial x^2} \text{ —— ②}$$

Given $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

Let $U = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ —— ③ and $V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ —— ④

Now differentiating eq^ns ③ and ④ partially with respect x.

$$\frac{\partial U}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} \quad — ⑤$$

$$\frac{\partial V}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \quad — ⑥$$

Differentiating eq^ns ③ and ④ partially with respect y.

$$\frac{\partial U}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} \quad — ⑦$$

$$\frac{\partial V}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 v}{\partial y^2} \quad — ⑧$$

From eq^n ① and ⑦

$$\frac{\partial U}{\partial y} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial U}{\partial y} = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right) = -\frac{\partial V}{\partial x} \quad [\because eq^n ⑥]$$

$$\therefore \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

From eq^n ② and ⑧

$$\frac{\partial V}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 v}{\partial x^2} \quad [\because$$

$$\frac{\partial V}{\partial y} = -\left[\frac{\partial^2 v}{\partial x^2} - \right]$$

$$\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x} \quad [\because eq^n ⑤]$$

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$\therefore$ C-R eq's are satisfied and the partial derivatives

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous.

$\therefore$ $U+iV = \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is analytic function.

4.Sol^n: Given $u(x,y) = 2\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = 2 \cdot \frac{1}{x^2+y^2} \cdot (2x) = \frac{4x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 4\left[(x^2+y^2)(1) - x(2x)\right] = 4x^2+4y^2-8x^2$$

$$\frac{\partial^2 u}{\partial x^2} = 4y^2 - 4x^2$$

$$\frac{\partial u}{\partial y} = 2 \cdot \frac{1}{x^2+y^2}(2y) = \frac{4y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = 4\left[(x^2+y^2)(1) - y(2y)\right] = 4x^2+4y^2-8y^2$$

$$\frac{\partial^2 u}{\partial y^2} = 4x^2 - 4y^2$$

To prove $u(x,y)$ is harmonic, we have to show

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow 4y^2 - 4x^2 + 4x^2 - 4y^2 = 0$

$\therefore$ $u(x,y)$ is harmonic.

To find conjugate of u :-

Let $u(x,y) = 2\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{4x}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{4y}{x^2+y^2}$$

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Let $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} + i\left(\frac{-\partial u}{\partial y}\right) \quad \left[\because \text{By C-R eq}^{n}\text{s } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}\right]$$

$$= (3x^2 - 3y^2)$$

$$f'(z) = \frac{4x}{x^2+y^2} + i\left(\frac{-4y}{x^2+y^2}\right)$$

By Milne-thomson method express $f'(z)$ completely interms of z, Replace x by z and y by 0.

$$\therefore f'(z) = \frac{4z}{z^2+0} + i(0)$$

$$f'(z) = \frac{4\cancel{z}}{z^{\cancel{2}}} = \frac{4}{z}$$

Now integrating we get $f(z) = \int f'(z)dz$

$$f(z) = 4\int \frac{1}{z}dz = 4\log z + c$$

Put $z = x + iy$

$$f(z) = 4\log(x+iy) + c$$

Complex Integration :-

Introduction :-

In this chapter we shall introduce the idea of line integral of a complex valid function $f(z)$ of a complex variable z. in a simple way. The theory of line integral together with the theory of power series and residues, constitutes a very important portion of the functions of the complex variable.

Let us consider $F(t) = u(t) + iv(t)$, $a \le t \le b$ where u and v are real valued, sectionally continuous, functions of t in closed interval $[a,b]$. each of these functions $u(t)$ and $v(t)$ is such that $[a,b]$ can be divided into a finite number of sub intervals in which each of which the functions are continuous. and as finite limits from the interior at both end points of each sub interval.

We define $\int_a^b F(t)dt = \int_a^b [u(t) + iv(t)]dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$

thus $\int_a^b F(t)dt$ is a complex number such that real part of $\int_a^b F(t)dt = \int_a^b u(t)dt$ and imaginary part of $\int_a^b F(t)dt = \int_a^b v(t)dt$.

Def:- A set of points (x, y) such that $x = x(t)$, $y = y(t)$, $a \le t \le b$ where $x(t)$, $y(t)$ are continuous functions of the real variable t is called a continuous arc. If no two distinct values of t correspond to the same point ordered pair (x, y) in the arc is called a "Jordan arc". 1

If $x(a) = x(b)$, $y(a) = y(b)$ and if no other two values of t correspond to the same point (x, y), the continuous arc is a simple closed curve. A simple closed curve is also called a Jordan curve.

- A contour is a continuous chain of a finite number of smooth arcs.

If a contour is closed and doesnot intersect itself, it is called a closed contour.

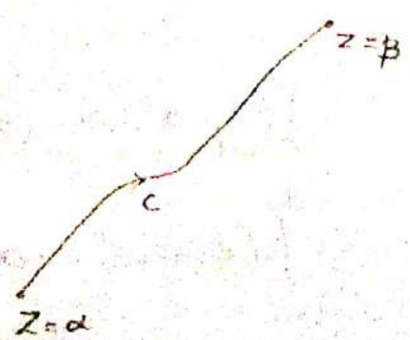
Line integral:-

Let $f(z)$ be a function of complex variable defined in a domain D.

Let 'c' be an arc in the domain joining from $z=\alpha$ to $z=\beta$. Let 'c' be defined by $x = x(t)$, $y = y(t)$ where $a \le t \le b$

where $\alpha = x(a) + iy(a)$ and

$\beta = x(b) + iy(b)$



Let $x(t)$, $y(t)$ be having continuous first order derivatives in $[a, b]$

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we define $\int_C f(z)dz = \int_{t=a}^{b} f[x(t)+iy(t)][\dot{x}(t)+i\dot{y}(t)]dt$

the above definition of $\oint_C f(z)dz$ can also be given in terms of limits of certain sum.

Note:-

Let $f(z)=u+iv$, where $z=x+iy \Rightarrow dz=dx+idy$

$$\therefore \boxed{\int_C f(z)dz = \int_C (u+iv)(dx+idy)}$$

Problems:-

1) Evaluate $\int (2y+x^2)dx+(3x-y)dy$ along the parabola $x=2t$, $y=t^2+3$ joining $(0,3)$ and $(2,4)$.

Sol^n: Given $\int (2y+x^2)dx+(3x-y)dy$

$$x=2t \Rightarrow dx=2dt$$

$$y=t^2+3 \Rightarrow dy=2tdt$$

$$x=0 \Rightarrow t=0$$

$$x=2 \Rightarrow t=1$$

$\therefore$ t from $t=0$ to $t=1$.

$$\therefore \int (2y+x^2)dx+(3x-y)dy$$

$$= \int_{t=0}^{1} \left(2(t^2+3)+(2t)^2\right)(2dt)+\left(3(2t)-(t^2+3)\right)(2tdt)$$

$$= \int_{t=0}^{1} [4t^2+12+8t^2+12t^2-2t^3-6t]dt$$

$$= \int_{t=0}^{1} [12-6t+24t^2-2t^3]dt$$

$$\left[12t - \frac{6t^2}{2} + \frac{24t^3}{3} - \frac{2t^4}{4}\right]_0^1$$

$$= 12[1-0] - 3[1-0] + 8[1-0] - \frac{1}{2}(1-0)$$

$$= 12-3+8-\frac{1}{2}$$

$$= \frac{33}{2}.$$

2) Evaluate $\int_0^{1+i} (x^2-iy)dz$ along the paths

i) $y=x$ ii) $y=x^2$

Sol^n: Given $\int_0^{1+i} (x^2-iy)dz$ i.e, $[(0,0)$ to$]$ $x+iy=0$ to $x+iy=1+i$

$(0,0)$ to $(1,1)$

i) Along the path $y=x$:-

$y=x \Rightarrow dy=dx$

x from $x=0$ to $x=1$

$$\int_0^{1+i} (x^2-iy)dz = \int_0^{1+i} (x^2-iy)(dx+idy) \qquad \left[\because z=x+iy \atop dz=dx+idy\right]$$

$$= \int_{x=0}^{1} (x^2-ix)(dx+idx)$$

$$= \int_{x=0}^{1} (x^2-ix)(1+i)dx$$

$$= (1+i)\left(\frac{x^3}{3} - i\frac{x^2}{2}\right)_0^1$$

$$= (1+i)\left(\frac{1}{3} - \frac{i}{2}\right)$$

$$= \frac{1}{3} - \frac{i}{2} + \frac{i}{3} - \frac{i^2}{2}$$

$$= \frac{1}{3} + \frac{1}{2} + i\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{5}{6} + i\left(\frac{-1}{6}\right)$$

ii) Along the path $y = x^2$:-

$y = x^2 \Rightarrow dy = 2x\,dx$

x from $x = 0$ to $x = 1$

$$\int_0^{1+i} (x^2 - iy)\,dz = \int_0^{1+i} (x^2 - iy)(dx + i\,dy) \quad \left[\because z = x + iy \atop dz = dx + i\,dy\right]$$

$$= \int_{x=0}^{1} (x^2 - ix^2)(dx + i\,2x\,dx)$$

$$= \int_{x=0}^{1} (x^2 - ix^2)(1 + i2x)\,dx$$

$$= \int_{x=0}^{1} (x^2 + i2x^3 - ix^2 + 2x^3)\,dx$$

$$= \left[\frac{x^3}{3} + i\cdot 2\cdot\frac{x^4}{4} - i\,\frac{x^3}{3} + 2\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} + i\cdot\frac{1}{2} - \frac{i}{3} + \frac{1}{2}$$

$= \frac{1}{3}+\frac{1}{2}+i\left(\frac{1}{2}-\frac{1}{3}\right)$

$= \frac{5}{6}+i\left(\frac{1}{6}\right)$

3) Evaluate $\int_0^{3+i} z^2 dz$ along

i) the line $y = x/3$ ii) parabola $x = 3y^2$

Sol: given $\int_0^{3+i} z^2 dz$ $z=0$ to $z=3+i$

i.e, $(0,0)$ to $(3,1)$

i) line $y = x/3$

$z = x+iy \Rightarrow dz = dx+idy$

$y = x/3 \Rightarrow dy = \frac{dx}{3}$

and $x=0$ to $x=3$

$$\int_0^{3+i} z^2 dz = \int_{x=0}^{3} (x+iy)^2 (dx+idy)$$

$$= \int_0^3 (x^2-y^2+2ixy)(dx+idy)$$

$$= \int_0^3 \left(x^2-\frac{x^2}{9}+2i\left(\frac{x}{3}\right)(x)\right)\left(dx+i\frac{dx}{3}\right)$$

$$= \left(\frac{x^3}{3}-\frac{x^3}{27}+2i\frac{x^3}{6}\right)_0^3+\frac{i}{3}\left(\frac{x^3}{3}-\frac{x^3}{27}+2i\frac{x^3}{6}\right)_0^3$$

$$= \left(3^2-\frac{27}{27}+i\,3^2\right)+\frac{i}{3}\left(3^2-1+i3^2\right)$$

$$= (9-1+9i)+\frac{i}{3}(9-1+9i) \Rightarrow (8+9i)+\frac{i}{3}(8+9i)$$

$$= 8-\frac{9}{3}+9i+\frac{8i}{3}$$

$$= 5+i\,\frac{35}{3}$$

4) Evaluate $\int_C (x-2y)dx + (y^2-x^2)dy$ where 'c' is the boundary of the first co-ordinate of the circle $x^2+y^2=4$.

Given

$$\int_C (x-2y)dx + (y^2-x^2)dy$$

where c is the boundary of the first quadrant of the circle $x^2+y^2=4$

Parametric eq^n of the circle $x = 2\cos\theta$, $y = 2\sin\theta$

$dx = 2(-\sin\theta)d\theta$ $\qquad$ $dy = 2\cos\theta d\theta$

and

$dx = -2\sin\theta d\theta$ $\qquad$ $0 \le \theta \le 2\pi$

Hence the given integral $\int_C (x-2y)dx + (y^2-x^2)dy$

$$= \int_0^{\pi/2} [(2\cos\theta - 2(2\sin\theta))(-2\sin\theta d\theta) + [(2^2\sin^2\theta - 2^2\cos^2\theta)(2\cos\theta d\theta)]$$

[$\because$ curve lies in first quadrant].

$$= \int_0^{\pi/2} \{[-4\sin\theta\cos\theta + 8\sin^2\theta] + [8\sin^2\theta\cos\theta - 8\cos^3\theta]\} d\theta$$

$$= \int_0^{\pi/2} -4\sin\theta\cos\theta d\theta + \int_0^{\pi/2} 8\sin^2\theta d\theta + \int_0^{\pi/2} 8\sin^2\theta\cos\theta d\theta - \int_0^{\pi/2} 8\cos^3\theta d\theta$$

$$= \int_0^{\pi/2} -2(2\sin\theta\cos\theta)d\theta + 8\int_0^{\pi/2} \sin^2\theta d\theta + 8\int_0^{\pi/2} \sin^2\theta\cos\theta d\theta - 8\int_0^{\pi/2} \cos^3\theta d\theta$$

$$= (-2)\int_0^{\pi/2} \sin 2\theta d\theta + 8\int_0^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta + 8\int_0^{\pi/2} \sin^2\theta\cos\theta d\theta - 8\int_0^{\pi/2} \cos^3\theta d\theta$$

$$= -2\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi/2} + 4\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} + 8\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2} - 8\int_0^{\pi/2} \frac{1}{4}[\cos 3\theta + 3\cos\theta] d\theta$$

$$\left[\because \int f(x)^n \cdot f'(x) = \frac{f(x)^{n+1}}{n+1}\right]$$

$$= \left[\cos 2\left(\tfrac{\pi}{2}\right) - 1\right] + 4\left[\tfrac{\pi}{2} - \tfrac{1}{2}\sin 2\left(\tfrac{\pi}{2}\right)\right] + \tfrac{8}{3}\left[\sin 3\left(\tfrac{\pi}{3}\right)\right]$$

$$- 2\left[\frac{\sin^3\theta}{3} + 3\sin\theta\right]_0^{\pi/2}$$

$$= [-1-1] + 4\left[\tfrac{\pi}{2}\right] + \tfrac{8}{3}(1) - 2\left[\frac{\sin\frac{3\pi}{2}}{3} + 3\sin\tfrac{\pi}{2}\right]$$

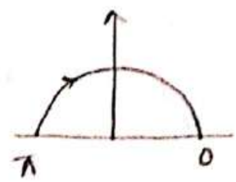
$$= -2 + 2\pi + \tfrac{8}{3} + \tfrac{2}{3} - 6$$

$$= 2\pi - \tfrac{14}{3}.$$

5) Evaluate $\int_C \bar{z}\,dz$ where 'c' is the upper half of the unit circle $|z| = 1$ taken in clock-wise sense.

-Ans: $(-\pi i)$

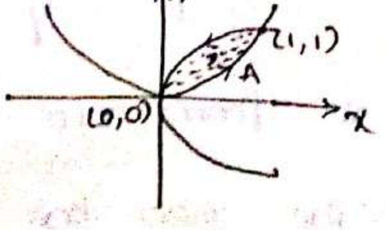
$\theta : \pi - 0$



6) Integrate $f(x) = x^2 + ixy$ from the point $A(1,1)$ to $B(2,8)$

i) Along the straight line AB ii) Along the curve $x = t$, $y = t^3$.

7) Evaluate $\int_C (y^2 + 2xy)dx + (x^2 - 2xy)dy$ where c is the boundary of the region $y = x^2$ and $x = y^2$



8) Evaluate $\int_{(0,1)}^{(1,2)} (x^2 - y)dx + (y^2 + x)dy$ along

i) A straight line from $(0,1)$ to $(1,2)$

ii) straight line from $(0,1)$ to $(1,1)$ and then from $(1,1)$ to $(1,2)$

iii) The parabola $x = t$, $y = t^2 + 1$

Given $\displaystyle\int_{(0,1)}^{(2,8)} (x^2-y)dx+(y^2+x)dy$

i) Along $(0,1)$ to $(1,2)$

Eqn of the line joining the two points $(0,1)$ & $(1,2)$ is $y=x+1$

then $y=x+1$

$dy=dx$

and x from 0 to 1.

$$\int_{(0,1)}^{(1,2)} (x^2-y)dx+(y^2+x)dy = \int_{x=0}^{1} (x^2-(x+1))dx+((x+1)^2+x)dx$$

$$= \int_{x=0}^{1} (x^2-\cancel{x}-\cancel{1}+x^2+2x+\cancel{1}+\cancel{x})dx$$

$$= \int_{x=0}^{1} (2x^2+2x)dx$$

$$= \frac{5}{3}$$

ii) Along the line $(0,1)$ to $(1,1)$:-

Along the straight line from $(0,1)$ to $(1,1)$ is

$y=1 \Rightarrow dy=0$

x from $x=0$ to $x=1$

$\therefore$ The given line integral becomes

$$\int_{(0,1)}^{(1,2)} (x^2-y)dx+(y^2+x)dy = \int_{x=0}^{1} (x^2-1)dx+(\ )(0)$$

$$= \int_{x=0}^{1} (x^2-1)dx$$

$$= \left(\frac{x^3}{3}-x\right)_0^1 = -\frac{2}{3}$$

$$\therefore \int_{(0,1)}^{(1,1)} (x^2-y)dx + (y^2+x)dy = \frac{-2}{3} \quad \text{—} ①$$

Now along the line (1,1) to (1,2):-

Along the straight line from (1,1) to (1,2) is $x=1 \Rightarrow dx=0$

y from $y=1$ to $y=2$

$$\therefore \int_{(1,1)}^{(1,2)} (x^2-y)dx + (y^2+x)dy = \int_{y=1}^{2} (1-y)(0) + (y^2+1)dy$$

$$= \frac{10}{3}$$

Required value $= -\frac{2}{3} + \frac{10}{3}$

$$= \frac{8}{3}$$

iii) Along the parabola $x=t$, $y=t^2+1$

$x=t$, $\Rightarrow dx = dt$

$y=t^2+1 \Rightarrow dy = 2t\,dt$ from $t=0$ to $t=1$.

Hence the given line integral becomes

$$\int_{(0,1)}^{(1,2)} (x^2-y)dx + (y^2+x)dy = \int_{t=0}^{1} (t^2-t^2-1)(dt) + ((t^2+1)+t)(2t\,dt)$$

$$= \int_{t=0}^{1} (-1+2t^5+2t+4t^3+2t^2)\,dt$$

$$= \int_{t=0}^{1} (2t^5+4t^3+2t^2+2t-1)dt$$

$= \underline{2}$

9) Evaluate $\int_C \bar{z}dz$ from $z=0$ to $4+2i$ along the curve c given by i) $z=t^2+it$ ii) Along the line $z=0$ to $z=2i$ and there from $z=2i$ to $4+2i$

The cauchy - Gaursat theorem:-

If a function $f(x)$ is analytic at all points interior to and on a simple closed curve 'c' then $\int_C f(z)dz=0$ This is called cauchy's Gaursat theorem.

State and prove cauchy's thm (cauchy's integral theorem)

Statement:-

Let $f(z)=u(x,y)+iv(x,y)$ be analytic on and with in a simple countour 'c' and let $f'(z)$ be continuous there then $\int_C f(z)dz=0$

Proof:-

we have $f(z)=u(x,y)+iv(x,y)$

let $z=x+iy$ then $dz=dx+idy$

$\therefore f(z)dz = (u+iv)(dx+idy)$

$= udx+iudy+ivdx-vdy$

$=(udx-vdy)+i(vdx+udy)$

$\int f(z)dz = \int_C (udx-vdy)+i(vdx+udy)$

$\int_C F(z)dz = \int_C (udx-vdy)+i\int_C (vdx+udy)$

By using green's thm, in a plane assume that 'R' is the region bounded by 'c' then $\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$

$\therefore$ Eqⁿ ① $\Rightarrow$ $\int_C F(z) dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx dy$

Since $f'(z)$ is continuous, the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region 'R' enclosed by 'c'.

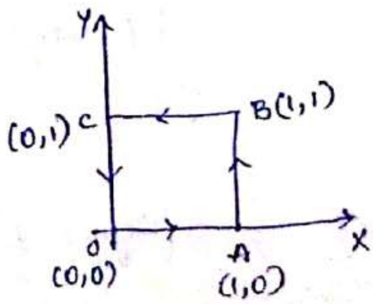
Given that $f(x) = u + iv$

is an analytic function, so by C-R eqⁿ

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\int_C f(z) dz = 0$$

1) Show that $\int_C (z+1) dz = 0$ where C is the boundary of the square whose vertices at the points $z=0, z=1, z=1+i, z=i$



Solⁿ: Let $I = \int_C (z+1) dz$

then $I = I_{OA} + I_{AB} + I_{BC} + I_{CO}$ —— ①

(i) Along] $I = \int_C (z+1) dz = \int_C (x+iy+1)(dx+idy)$ $\left[\because z = x+iy \right.$
$\left. dz = dx + idy \right]$

$= \int_C [(x+1) + iy](dx+idy)$

$$\therefore I = \int_C ((x+1)+iy)(dx+idy) \quad \text{—— (2)}$$

i) Along OA :-

OA is $y=0 \Rightarrow dy=0$

x from $x=0$ to $x=1$

$$\therefore (2) \Rightarrow I_{OA} = \int_{x=0}^{1} (x+1)dx$$

$$= \left(\frac{x^2}{2}+x\right)_0^1$$

$$= \frac{1}{2}+1$$

$$I_{OA} = \frac{3}{2} \quad \text{—— (3)}$$

ii) Along AB :-

AB is $x=1 \Rightarrow dx=0$

y from 0 to 1.

$$(2) \Rightarrow I_{AB} = \int_{y=0}^{1} (1+1+iy)(idy)$$

$$= \int_{y=0}^{1} (2+iy)(idy)$$

$$= i\left[2y+i\frac{y^2}{2}\right]_0^1$$

$$= i\left[2+\frac{i}{2}\right]$$

$$\therefore I_{AB} = 2i-\frac{1}{2} \quad \text{—— (4)}$$

iii) Along BC :-

BC is $y=1 \Rightarrow dy=0$

x from $x=1$ to $x=0$

$$(2) \Rightarrow I_{BC} = \int_{x=1}^{0} ((x+1)+i)dx$$

$$= \left(\frac{x^2}{2}+x+ix\right)_0^1$$

$$= (0)-\left(\frac{1}{2}+1+i\right)$$

$$I_{BC} = -\frac{3}{2}-i \quad \text{—— (5)}$$

iv) Along CO:-

CO is $x=0 \Rightarrow dx=0$

y from $y=1$ to $y=0$

$$(2) \Rightarrow I_{CO} = \int_{y=1}^{0} (1+iy)(idy)$$

$$= i\left(y+i\frac{y^2}{2}\right)_1^0$$

$$= i\left[(0)-\left(1+\frac{i}{2}\right)\right]$$

$$I_{CO} = -i+\frac{1}{2} \quad \text{—— (6)}$$

Now sub eq^{n}s (3), (4), (5), (6) in eq^n(1), we get

$$(1) \Rightarrow I = \cancel{\frac{3}{2}}+2i-\cancel{\frac{1}{2}}-\cancel{\frac{3}{2}}-i-i+\cancel{\frac{1}{2}}$$

$$= 2i-2i$$

$$= 0$$

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$\therefore \int_C (z+1)dz = 0$

2) If c is the boundary of the square with vertices at the points $z=0$, $z=1$, $z=1+i$, $z=i$. Show that $\int_C (3z+1)dz = 0$

Note:-

To Evaluate $\int_C f(z)dz$ where c is any simple closed contour, plot the curve c, locate the singular point of $f(z)$, if none is on and within c, $\int_C f(z)dz = 0$ by cauchy's integral theorem. ~~If~~

If some singular points are within 'c', we shall develop methods for evaluation of $\int_C f(z)dz$ subsequently.

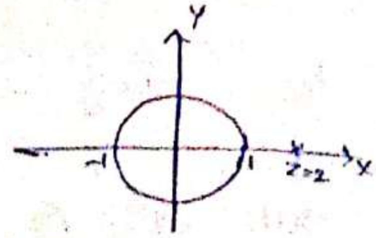
1) Evaluate $\int_C \frac{e^{2z}}{z-2}dz$ where c is $|z|=1$

Given $\int_C \frac{e^{2z}}{z-2}dz$, $c: |z|=1$

$$\int_C f(z)dz = \int_C \frac{e^{2z}}{z-2}dz$$

$$f(z) = \frac{e^{2z}}{z-2}$$

$z=2$ is singular point of $f(z)$



The singular point $z=2$ is outside the curve $|z|=1$

$\therefore$ By cauchy's integral theorem

$$\int_C \frac{e^{2z}}{z-2}dz = 0$$

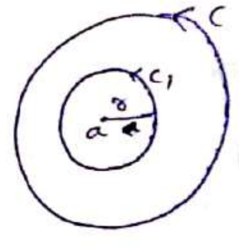
State and prove Cauchy's Integral formula.

If $f(z)$ is analytic on and within a closed contour C. If $z=a$ is any point within C then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}\,dz \quad \text{i.e.,} \quad \int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \left[2\pi i (f(z))_{\text{at } z=a}\right]$$

Proof:-

If $f(z)$ be analytic within and on a simple closed curve 'c', $z=a$, a point within 'c'.



we prove that $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

Draw a small circle c_1 with 'a' as centre & radius r, so small such that c_1 lies inside C.

Then $\frac{f(z)}{z-a}$ is analytic within 'c' except $z=a$

By using Cauchy's thm for multiply connected domain

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \quad —① $$

c_1 is the circle with centre at a & radius r.

Eqn of circle c_1 is $|z-a| = r$

$$z - a = re^{i\theta}$$

$$z = a + re^{i\theta}$$

$$dz = ire^{i\theta}\,d\theta$$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{re^{i\theta}} \cdot ire^{i\theta}\,d\theta$$

$$= \int_{C_1} f(z) \cdot i\,d\theta$$

$$= i\int_{C_1} f(z)\,d\theta$$

$$= i\int_{C_1} f(a + re^{i\theta})\,d\theta$$

As $r \to 0$ the circle 'C' shrinks to the point a & $f(a+re^{i\theta}) \to f(a)$

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{C_1} f(a)\,d\theta = if(a)\int_{C_1} d\theta$$

$$= if(a)\int_0^{2\pi} d\theta$$

$$= if(a)\,[\theta]_0^{2\pi}$$

$$= if(a)[2\pi - 0]$$

$$= 2\pi i f(a)$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.$$

State and prove generalization of cauchy's integral formula:-

Statement:-

If $f(z)$ is analytic on and within a simple closed curve 'c' and if 'a' is any point within 'c' then $f^{(n)}(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz$

i.e, $\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$.

Problems:-

1) Evaluate $\oint_C \frac{z^2+4}{z-3}\,dz$ where c is i) $|z|=5$ and ii) $|z|=2$.

Sol^n: Given $\oint_C \frac{z^2+4}{z-3}\,dz$

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_C \frac{z^2+4}{z-3}\,dz$$

$$f(z)=z^2+4 \quad , \quad a=3$$

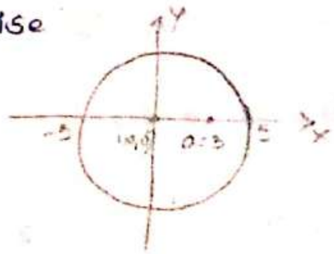
i) $c : |z|=5$

Let $z=x+iy$ then

$|x+iy|=5 \Rightarrow x^2+y^2=5^2$

is the circle with centre at $(0,0)$ and radius 5 units.

The curve $c: |z|=5$ taken in anti-clockwise sense.



The singular point $a=3$ is lies inside the contour c, $|z|=5$.

The function $f(z)$ is analytic except at $z=3$

By cauchy's integral formula $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$

$$\Rightarrow \int_C \frac{f(z)}{z-a}\,dz = 2\pi i\,[f(z)]_{z=a}$$

$$\int_C \frac{z^2+4}{z-3} = 2\pi i\,[z^2+4]_{z=3}$$

$$= 2\pi i\,[3^2+4] = 26\pi i$$

$$\oint_C \frac{z^2+4}{z-3} = 26\pi i$$

ii) $|z| = 2$

$|z| = 2$ is a circle centre at $(0,0)$ and radius 2 units.

The singular point $z = 3$ is outside the curve. Hence by cauchy's thm $\oint_C \frac{z^2+4}{z-3}\, dz = 0$.

2) Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$

Soln: Given $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$, $|z| = 3$

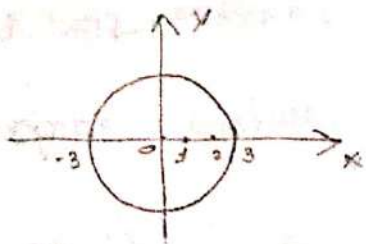
Now $\frac{1}{(z-1)(z-2)} = \left[\frac{1}{z-2} - \frac{1}{z-1}\right]$ [$\because$ By partial fractions]

$\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)} dz = \int_C \frac{e^{2z}}{z-2} dz - \int \frac{e^{2z}}{z-1} dz$

Given c is $|z| = 3$, is the circle with centre at $(0,0)$ and radius 3 units.

The points $z = 1, 2$ are within the circle $c; |z| = 3$

By cauchy's Integral formula



$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i [f(z)]_{z=a}$$

$$① \Rightarrow \int_C \frac{e^{2z}}{(z-1)(z-2)} dz = 2\pi i \left[e^{2z}\right]_{z=2} - 2\pi i \left[e^{2z}\right]_{z=1}$$

$$= 2\pi i [e^4] - 2\pi i [e^2]$$

$$= 2\pi i (e^4 - e^2).$$

3) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$ around $c: |z-1| = 3$

Soln: Given $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \int \frac{e^{2z}}{(z+1)^4}\,dz$$

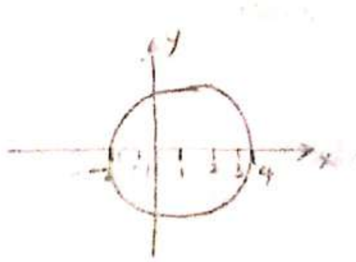
$f(z) = e^{2z}$, $n = 3$, $a = -1$

$c: |z-1| = 3$ is the circle at centre $z = 1$ and radius 3 units

$\frac{e^{2z}}{(z+1)^4}$ has $z = -1$ as its singular point and it is within c

By generalised cauchy's Integral formula

$[-3 \le z-1 \le 3$
$-2 \le z \le 4]$



$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a) = \frac{2\pi i}{n!} \left(f^n(z)\right)_{z=a}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!} \left(f'''(z)\right)_{z=-1}$$

$$= \frac{2\pi i}{6} \left[\frac{d^3}{dz^3}\left(e^{2z}\right)\right]_{z=-1}$$

$$= \frac{\pi i}{3} \left[\frac{d^2}{dz^2}\left(2e^{2z}\right)\right]_{z=-1}$$

$$= \frac{\pi i}{3} \left[\frac{d}{dz}\left(4e^{2z}\right)\right]_{z=-1}$$

$$= \frac{\pi i}{3} \left[8e^{2z}\right]_{z=-1}$$

$$= \frac{\pi i}{3} \left[8e^{-2}\right]$$

$$= \frac{8\pi i}{3} e^{-2}$$

$$\therefore \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i}{3} e^{-2}.$$

4)-Evaluate

i) $\int_C \frac{e^z}{(z-1)(z-4)} dz$, where $c: |z| = 2$

ii) $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)^3} dz$, $c: |z| = 3$

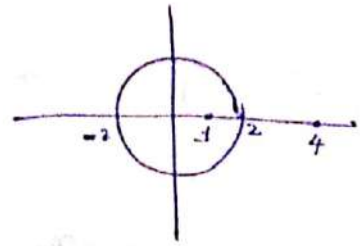
Sol^n: i) Given $\int_C \frac{e^z}{(z-1)(z-4)} dz$, $|z| = 2$

Now $\frac{1}{(z-1)(z-4)} = \frac{-1}{3(z-1)} + \frac{1}{3(z-4)}$

$$\therefore \int_C \frac{e^z}{(z-1)(z-4)} dz = \int \frac{-e^z}{3(z-1)} dz + \int \frac{e^z}{3(z-4)} dz$$

Given c is $|z| = 2$, is the circle with centre at $(0,0)$ & radius 2 units.

The point $z=1$ is within the circle and point $z = 4$ is outside the circle.



By cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i \left[f(z)\right]_{z=a}$$

$$\int_C \frac{e^z}{(z-1)(z-4)} dz = -\frac{1}{3} \int \frac{e^z}{z-1} dz + \frac{1}{3} \int \frac{e^z}{z-4} dz$$

$$= -\frac{1}{3} \left[2\pi i (e^z)_{z=1}\right] + \frac{1}{3}(0) \quad [\because \text{outside}]$$

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$$= -\frac{1}{3}\left[2\pi i\,(e^{1})\right]$$

$$\int_C \frac{e^z}{(z-1)(z-4)}\,dz = -\frac{2\pi i e}{3}$$

ii) Given $\displaystyle\int_C \frac{\cos\pi z^2}{(z-1)(z-2)^3}$, $C: |z| = 3$

$$\frac{1}{(z-1)(z-2)^3} = \frac{-1}{z-1} + \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3}$$

Given C is $|z| = 3$, is the circle with centre at $(0,0)$ & radius 3 units.

The points $z=1$, $z=2$ lies within the circle.

By cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\left[f(z)\right]_{z=a} \quad \text{and}$$

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}\left[f^n(a)\right]_{z=a}$$

$$\int_C \frac{\cos\pi z^2}{(z-1)(z-2)^3} = \int \frac{-\cos\pi z^2}{z-1}\,dz + \int \frac{\cos\pi z^2}{(z-2)}\,dz - \int \frac{\cos\pi z^2}{(z-2)^2}\,dz + \int \frac{\cos\pi z^2}{(z-2)^3}\,dz$$

$$= -2\pi i\left[\cos\pi z^2\right]_{z=1} + 2\pi i\left[\cos\pi z^2\right]_{z=2} - \frac{2\pi i}{1!}\left[f'(z)\right]_{z=2} +$$

$$\frac{2\pi i}{2!}\left[f''(z)\right]_{z=2}$$

$= -2\pi i\,[\cos\pi] + 2\pi i\,[\cos 4\pi] - 2\pi i\,[$

5) Evaluate $\int_C \frac{dz}{(z-2i)^2(z+2i)^2}$, C be the circumference of the ellipse $x^2+4(y-2)^2=4$.

Soln: Given ellipse $x^2+4(y-2)^2=4$

$$\frac{(x-0)^2}{4}+\frac{(y-2)^2}{1}=1$$

$$\frac{(x-0)^2}{2^2}+\frac{(y-2)^2}{1^2}=1$$

It's centre is $(0,2)$

The integrant $\int \frac{1}{(z-2i)^2(z+2i)^2}$ has two singular points $z=2i$, $z=-2i$. of which only $z=2i$ lies inside 'c'.

Now consider $f(z)=\frac{1}{(z+2i)^2}$

So, $$\int_C \frac{dz}{(z-2i)^2(z+2i)^2} = \int_C \frac{\left[1/(z+2i)^2\right]}{(z-2i)^2}dz = \int_C \frac{f(z)}{(z-a)^{1+1}}dz$$

Let $z=2i$ and $n=1$

By using generalised cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^n(a) = \frac{2\pi i}{n!}\left(f^{(n)}(z)\right)_{z=a}$$

$$\therefore \int_C \frac{dz}{(z-2i)^2(z+2i)^2} = \frac{2\pi i}{1!}\left[f'(z)\right]_{z=2i}$$

$$= 2\pi i \left[\frac{d}{dz} \left(\frac{1}{(z+2i)^2} \right) \right]_{z=2i}$$

$$= 2\pi i \left[\frac{-2}{(z+2i)^3} \right]_{z=2i}$$

$$= -4\pi i \left[\frac{1}{(2i+2i)^3} \right]$$

$$= -4\pi i \left[\frac{1}{-64 i} \right]$$

16

$$= \frac{\pi}{16}$$

$$\therefore \int_C \frac{dz}{(z-2i)^2(z+2i)^2} = \frac{\pi}{16}.$$

6) Evaluate $\int_C \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^3}\, dz$ with c, $|z| = 2$ using cauchy's integral formula.

Soln: $\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \int_C \frac{z^3 - \sin 3z}{(z-\frac{\pi}{2})^{2+1}}\, dz$

$f(z) = z^3 - \sin 3z$, $a = \pi/2$ & $n = 2$

The singular point of the integrand is $\pi/2$, $z = \pi/2$ lies inside c

$\therefore$ By Generalised cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} \left(f^{(n)}(a) \right)$$

$$\therefore \int_C \frac{z^3 - \sin 3z}{(z-\pi/2)^3}\, dz = \frac{2\pi i}{2!} \left[f''(z^3 - \sin 3z) \right]_{z=\pi/2}$$

$$= \frac{2\pi i}{2}\left[\frac{d}{dz}\left[3z^2 - 3\cos 3z\right]\right]_{z=\pi/2}$$

$$= \pi i\left[6z + 9\sin 3z\right]_{z=\pi/2}$$

$$= \pi i\left[\overset{3}{\cancel{6}}\cdot\frac{\pi}{\cancel{2}} + 9\sin 3\cdot\frac{\pi}{2}\right]$$

$$= \pi i\left[3\pi - 9\right]$$

$$= 3\pi i\left[\pi - 3\right]$$

7). Evaluate $\oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz$, if C is the circle $|z| = 1$.

Soln: $\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^{2+1}}\,dz$

i.e, $f(z) = \sin^2 z$, $a = \pi/6$ & $n = 2$

The singular point of the integrand is $\pi/6$, $z = \pi/6$ is lies inside $C : |z| = 1$.

By Generalised cauchy's integral formula.

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}\left(f^n(a)\right)$$

$$\oint_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz = \frac{\cancel{2}\pi i}{\cancel{2!}}\left[f''(\sin^2 z)\right]_{z=\frac{\pi}{6}}$$

$$= \pi i\left[\frac{d}{dz}\left(2\sin z\cos z\right)\right]_{z=\frac{\pi}{6}}$$

$$= \pi i\left[\frac{d}{dz}\left(\sin 2z\right)\right]_{z=\frac{\pi}{6}}$$

$$= \pi i\left[2\cos 2z\right]_{z=\frac{\pi}{6}}$$

$$= \pi i \left[2\cos\left(2\left(\frac{\pi}{6}\right)\right)\right]$$

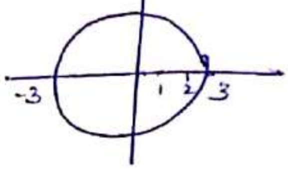
$$= \pi i \left[2\cos\frac{\pi}{3}\right]$$

$$= \pi i \left[\cancel{2}\cdot\frac{1}{\cancel{2}}\right]$$

$$= \pi i$$

8)-Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$, where c is the circle $|z| = 3$ using cauchy's integral formula.

Soln: $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$, $c: |z| = 3$



Now, $\frac{1}{(z-1)(z-2)} = \left[\frac{1}{z-2} - \frac{1}{z-1}\right]$

The integrand has two singular points $z=1, z=2$

The circle c is $|z| = 3$ is the circle centre at $(0,0)$ & radius 3 units.

The two singular points $z = 1, 2$ lies inside $c: |z| = 3$.

$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = \int \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz - \int \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz$$

$$= 2\pi i \left[\left[\sin \pi z^2 + \cos \pi z^2\right]_{z=2} - \left[\sin \pi z^2 + \cos \pi z^2\right]_{z=1}\right]$$

$$= 2\pi i \left[\sin 4\pi + \cos 4\pi - \sin \pi - \cos \pi\right]$$

$$= 2\pi i \left[1 - (-1)\right]$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = 4\pi i$$

9) Evaluate using cauchy's thm $\int_C \frac{z^3 e^{-z}}{(z-1)^3} dz$ where $C : |z-1| = 1/2$.

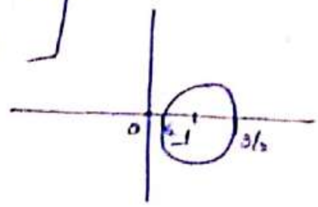
Sol^n: $\int_C \frac{z^3 e^{-z}}{(z-1)^3} dz = \int_C \frac{f(z)}{(z-a)^{n+1}} dz$, $C : |z-1| = 1/2$

The integrand has one singular point $z = 1$.

The given curve $C : |z-1| = \frac{1}{2}$ is the circle at centre with $z = 1$.

$$\left[\begin{array}{c} -\frac{1}{2} < z-1 < \frac{1}{2} \\ \frac{1}{2} < z < \frac{3}{2} \end{array}\right]$$

The point $z = 1$ lies inside the C.
$|z-1| = 1/2$



consider $f(z) = z^3 e^{-z}$, $a = 1$, $n = 2$

By cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} \left(f^n(z)\right)_{z=a}$$

$$\int_C \frac{e^{-z} z^3}{(z-1)^{2+1}} dz = \frac{2\pi i}{2!} \left[f''(z^3 e^{-z})\right]_{z=1}$$

$$= \pi i \left[\frac{d}{dz}\left(3z^2 e^{-z} - e^{-z} z^3\right)\right]_{z=1}$$

$$= \pi i \left[6ze^{-z} - 3z^2 e^{-z} - 3z^2 e^{-z} + z^3 e^{-z}\right]_{z=1}$$

$$= \pi i \left[\cancel{6e^{-1}} - \cancel{3e^{-1}} - \cancel{3e^{-1}} + e^{-1}\right]$$

$$= \pi i \left[\frac{1}{e}\right]$$

$$\therefore \int_C \frac{e^{-z} z^3}{(z-1)^3} dz = \frac{\pi i}{e}$$

10) Evaluate using cauchy's integral formula $\int_C \frac{z+1}{(z^2+2z+4)} dz$ where

$c : |z+1+i| = 2$.

Solⁿ: Given $\int_C \frac{z+1}{(z^2+2z+4)} dz$, $c : |z+1+i| = 2$

$$c : |z+1+i| = 2$$

$$\Rightarrow |z-(-1-i)| = 2$$

This is a circle with centre $(-1,-i)$ and radius is 2 units

Let $f(z) = \frac{z+1}{z^2+2z+4} = \frac{z+1}{(z+1)^2+3}$

$$= \frac{z+1}{(z+1+i\sqrt{3})(z+1-i\sqrt{3})}$$

$$= \frac{(z+1)}{(z-(-1-i\sqrt{3}))(z-(-1+i\sqrt{3}))}$$

The singular points are $z=-1-i\sqrt{3}$ & $z=-1+i\sqrt{3}$

Now consider $f(z) = \frac{(z+1)}{z-(-1+i\sqrt{3})}$

This function is analytic at all points inside c in fact that it is analytic everywhere except $z = -1+i\sqrt{3}$

The point $z=-1-i\sqrt{3}$ is only lies inside c

Hence by cauchy's integral formula

$$\int_C \frac{f(z)dz}{z-a} = 2\pi i\, f(a)$$

$$\therefore \int_C \frac{((z+1)/[z-(-1+i\sqrt{3})])}{(z-(-1-i\sqrt{3}))} = 2\pi i \left[\frac{z+1}{z-(-1+i\sqrt{3})}\right]_{z=(-1-i\sqrt{3})}$$

$$= 2\pi i \left[\frac{\cancel{-1}-i\sqrt{3}+\cancel{1}}{\cancel{-1}-i\sqrt{3}+\cancel{1}-i\sqrt{3}}\right]$$

$$= 2\pi i \left[\frac{-i\sqrt{3}}{-2i\sqrt{3}}\right]$$

$$= \pi i$$

$$\therefore \int_C \frac{z+1}{(z^2+2z+4)}\, dz = \pi i.$$

11) Evaluate using cauchy's integral formula $\int_C \frac{z+1}{(z^3-4z)}\, dz$ where

$C : |z+2| = \frac{3}{2}$.

Soln: $\int_C \frac{z+1}{(z^3-4z)}\, dz = \int_C \frac{z+1}{z(z^2-4)}\, dz = \int_C \frac{(z+1)}{z(z+2)(z-2)}\, dz$

$$\left[\begin{array}{c} -\frac{3}{2} < z+2 < \frac{3}{2} \\ -7/2 < z < \frac{-1}{2} \end{array}\right]$$

$C : |z+2| = \frac{3}{2}$

is the centre (centre) at $z=2$ & radius is 1.5 units.

The integrand has 3 singular points $z=0, z=2, z=-2$

out of these singular points only the point $z=-2$ lies inside the 'c'.

Let $f(z) = \frac{z+1}{z(z-2)}$

By cauchy's integral formula

$$\int_C \frac{z+1}{z^3-4z}\, dz = \int_C \frac{[(z+1)/z(z-2)]}{(z+2)}$$

$$= 2\pi i \left[\frac{z+1}{z(z-2)}\right]_{z=-2}$$

Problems:

1. Find the Laurent's series expansion of the function

$\dfrac{z^2-1}{(z+2)(z+3)}$ if $2<|z|<3$

Soln: Given $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$, $2<|z|<3$

$$f(z)= 1-\frac{5z+7}{(z+2)(z+3)}$$

$$z^2+5z+6\,)\,z^2-1\,(\,1$$
$$\underline{z^2+5z+6}$$
$$-5z-7$$

$$f(z)= 1-\left(\frac{(-3)}{z+2}+\frac{8}{z+3}\right)$$

$$f(z)= 1+\frac{3}{z+2}-\frac{8}{z+3}$$

Given, $2<|z|<3$ i.e., $\left|\frac{2}{z}\right|<1$ & $\left|\frac{z}{3}\right|<1$]

$$= 1+\frac{3}{z\left[1+\frac{2}{z}\right]}-\frac{8}{3\left[1+\frac{z}{3}\right]}$$

$$= 1+\frac{3}{z}\left[1+\frac{2}{z}\right]^{-1}-\frac{8}{3}\left[1+\frac{z}{3}\right]^{-1}$$

$$= 1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

$$= \left(1+\frac{3}{z}-\frac{6}{z^2}+\frac{12}{z^3}-\cdots\right)-\frac{8}{3}+\frac{8}{9}z-\frac{8}{27}z^2+\cdots$$

$$= \left(1-\frac{8}{3}\right)+\frac{3}{z}-\frac{6}{z^2}+\frac{12}{z^3}-\cdots+\frac{8}{9}z-\frac{8}{27}z^2+\cdots$$

$$= -\frac{5}{3} + \frac{3}{z} - \frac{6}{z^2} + \frac{12}{z^3} + \cdots + \frac{8z}{9} - \frac{8z^2}{27} + \cdots$$

$\therefore$ This is the laurent's series expansion of $f(z)$.

2. Expand $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ in powers of 'z', where

i) $|z|<1$ ii) $1<|z|<2$ iii) $|z|>2$

(or)

Expand $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ in powers of 'z',

i) within the unit circle about the origin [having radii of 1 & 2 res]

ii) within the angular region b/w the concentric circles about the origin having radii 1 & 2 respectively and

iii) The exterior to the circle of radius 2.

Given $f(z) = \dfrac{z+3}{z(z^2-z-2)}$

$$= \frac{z+3}{z(z-2)(z+1)}$$

$$\frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1} = \frac{z+3}{z(z-2)(z+1)}$$

$$A(z-2)(z+1) + B(z)(z+1) + C(z)(z-2) = z+3$$

By resolving into partial fractions

$$f(z) = -\frac{3}{2}\cdot\frac{1}{z} + \frac{2}{3}\cdot\frac{1}{z+1} + \frac{5}{6}\cdot\frac{1}{z-2}$$

i) $|z|<1$:-

$$f(z) = -\frac{3}{2}\cdot\frac{1}{z} + \frac{2}{3}\cdot\frac{1}{z+1} + \frac{5}{6}\cdot\frac{1}{-2\left[1-\frac{z}{2}\right]}$$

Notice, that both $\left|\frac{z}{2}\right|$ and $|z|$ are less than 1 $\left|\frac{z}{2}\right|$ & $|z|<1$

$$\therefore f(z) = -\frac{3}{2}\cdot\frac{1}{z} + \frac{2}{3}\left[1+z\right]^{-1} - \frac{5}{12}\left[1-\frac{z}{2}\right]^{-1}$$

$$= \frac{-3}{2}\cdot\frac{1}{z} + \frac{2}{3}\left[1 - z + z^2 - z^3 + z^4 - \cdots\right] + -\frac{5}{12}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$= \frac{-3}{2}\cdot\frac{1}{z} + \left(\frac{2}{3} - \frac{5}{12}\right) + \left(\frac{-2}{3} - \frac{5}{24}\right)z + \left(\frac{2}{3} - \frac{5}{48}\right)z^2 + \left(\frac{-2}{3} - \frac{5}{96}\right)z^3 + \cdots$$

$$\therefore f(z) = \frac{-3}{2}\cdot\frac{1}{z} + \frac{1}{4} - \frac{7}{8}z + \frac{9}{16}z^2 - \frac{2^3}{32}z^3 + \cdots$$

[for $|z| < 1$]

ii) $1 < |z| < 2$:-

$1 < |z| < 2$

i.e., $\left|\frac{1}{z}\right| < 1$ & $\left|\frac{z}{2}\right| < 1$

$$\therefore f(z) = \frac{-3}{2}\cdot\frac{1}{z} + \frac{2}{3}\cdot\frac{1}{z+1} + \frac{5}{6}\cdot\frac{1}{z-2}$$

$$= \frac{-3}{2}\cdot\frac{1}{z} + \frac{2}{3}\cdot\frac{1}{z\left(1+\frac{1}{z}\right)} + \frac{5}{6}\cdot\frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$f(z) = \frac{-3}{2}\cdot\frac{1}{z} + \frac{2}{3z}\left[1+\frac{1}{z}\right]^{-1} - \frac{5}{12}\left[1-\frac{z}{2}\right]^{-1}$$

$$= \frac{-3}{2}\cdot\frac{1}{z} + \frac{2}{3z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right] - \frac{5}{12}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$= \left[\left(\frac{-3}{2z} + \frac{3}{2z}\right) - \frac{2}{3}\cdot\frac{1}{z^2} + \frac{2}{3}\cdot\frac{1}{z^3} - \frac{2}{3}\cdot\frac{1}{z^4} + \cdots\right] - \frac{5}{12} - \frac{5}{12}\left[\frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right]$$

$$f(z) = \frac{-5}{12} - \frac{5}{6z} + \frac{2}{3}\left[-\frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\right] - \frac{5}{12}\left[\frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right]$$

iii) $|z| > 2$

i.e, $\left|\frac{z}{2}\right| > 1$ i.e, $\left|\frac{2}{z}\right| < 1$

$$f(z) = \frac{-3}{2}\cdot\frac{1}{z} + \frac{2}{3}\left(\frac{1}{z+1}\right) + \frac{5}{6}\cdot\frac{1}{z-2}$$

$$= \frac{-3}{2}\cdot\frac{1}{z} + \frac{2}{3}\,\frac{1}{z\left(1+\frac{1}{z}\right)} + \frac{5}{6}\cdot\frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$f(z) = \frac{-3}{2}\cdot\frac{1}{z} + \frac{2}{3z}\left[1+\frac{1}{z}\right]^{-1} + \frac{5}{6z}\left[1-\frac{2}{z}\right]^{-1}$$

$$= -\frac{3}{2}\cdot\frac{1}{z} + \frac{2}{3z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right] + \frac{5}{6z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$

$$= \left[\left(\frac{-3}{2z}+\frac{2}{3z}\right) - \frac{2}{3}\cdot\frac{1}{z^2}+\frac{2}{3}\cdot\frac{1}{z^3}-\frac{2}{3}\cdot\frac{1}{z^4}+\cdots\right] + \frac{5}{6z}+\frac{5}{3z^2}+\frac{10}{3z^3}+\frac{20}{3z^4}+\cdots$$

$$= -\frac{\cancel{5}}{\cancel{6}z} + \frac{2}{3}\left[-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right] + \frac{\cancel{5}}{\cancel{6}z}+\frac{5}{3z^2}+\frac{10}{3z^3}+\frac{20}{3z^4}+\cdots$$

$$= \frac{2}{3}\left[-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right]+\frac{5}{3z^2}+\frac{10}{3z^3}+\frac{20}{3z^4}+\cdots$$

$$= \left(\frac{5}{3}-\frac{2}{3}\right)\frac{1}{z^2}+\left(\frac{10}{3}+\frac{2}{3}\right)\frac{1}{z^3}+\left(\frac{20}{3}-\frac{2}{3}\right)\frac{1}{z^4}+\cdots$$

$$= \frac{1}{z^2}+4\frac{1}{z^3}+6\cdot\frac{1}{z^4}+\cdots \qquad \therefore f(z) = \frac{1}{z^2}+\frac{4}{z^3}+\frac{6}{z^4}+\cdots$$

3. Expand $\frac{7z-2}{z(z+1)(z-2)}$ about the point $z=-1$ in the region $1<|z+1|<3$.

Sol^n: Given $f(z) = \frac{7z-2}{z(z+1)(z-2)}$

$$\frac{A}{z}+\frac{B}{z+1}+\frac{C}{z-2} = \frac{7z-2}{z(z+1)(z-2)}$$

$$A(z+1)(z-2)+Bz(z-2)+C(z+1)z = 7z-2$$

$$f(z) = \frac{-3}{z+1}+\frac{1}{z}+\frac{2}{z-2}$$

Given $1<|z+1|<3$

i.e., $\left|\frac{1}{z+1}\right|<1$ and $\left|\frac{z+1}{3}\right|<1$

$$f(z) = \frac{-3}{z+1}+\frac{1}{(z+1)-1}+\frac{2}{(z+1)-3}$$

$$= \frac{-3}{z+1}+\frac{1}{(z+1)\left[1-\frac{1}{z+1}\right]} - \frac{2}{3\left[1-\left(\frac{z+1}{3}\right)\right]}$$

$$= \frac{-3}{z+1}+\frac{1}{(z+1)}\left[1-\frac{1}{z+1}\right]^{-1}-\frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}$$

$$= \frac{-3}{z+1}+\left(\frac{1}{z+1}\right)\left[1+\frac{1}{z+1}+\left(\frac{1}{z+1}\right)^2+\left(\frac{1}{z+1}\right)^3+\cdots\right]-\frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

$$= \left[\frac{-3}{z+1}+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\frac{1}{(z+1)^4}+\cdots\right]-\frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

$$f(z) = \left[\left(\frac{-2}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\left(\frac{1}{z+1}\right)^3+\left(\frac{1}{z+1}\right)^4+\cdots\right]-\frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

$$\left[\because \left|\frac{1}{z+1}\right|<1 \ \& \ \left|\frac{z+1}{3}\right|<1\right]$$

Determine the Lorentz's series expansion $f(z) = \frac{1}{z^2-4z+3}$ for

i) $1<|z|<3$ ii) $|z|<1$ iii) $|z|>3$.

Sol^n: Given function $f(z) = \frac{1}{z^2-4z+3}$

$$\frac{1}{z^2-4z+3} = \frac{1}{z^2\left(1-\left(\frac{4z-3}{z^2}\right)\right)}$$

$$f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$\frac{1}{z^2-4z+3} = \frac{Ax+B}{}$$

case(i): $1<|z|<3$

$1<|z|<3$ i.e $\left|\frac{1}{z}\right|<1$ & $\left|\frac{z}{3}\right|<1$

$$f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{z\left[1-\frac{1}{z}\right]}\right] \quad \left[\because \left|\frac{1}{z}\right|<1 \text{ \& } \left|\frac{z}{3}\right|<1\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3}\left[1-\frac{z}{3}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}\right]$$

$$= \frac{-1}{6}\left[1-\frac{z}{3}\right]^{-1} - \frac{1}{2z}\left[1-\frac{1}{z}\right]^{-1}$$

$$= \frac{-1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\dots\right] - \frac{1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right]$$

$$f(z) = -\frac{1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\dots\right] - \frac{1}{2}\left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right] \quad \left(\because \left|\frac{1}{z}\right|<1 \text{ \& } \left|\frac{z}{3}\right|<1\right)$$

which is the Laurent series expansion of $f(z)$ for $1<|z|<3$

ii) $|z|<1$

Given $f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$

$|z|<1$ $\left[\text{i.e., } |z|<1 \text{ \& } \left|\frac{z}{3}\right|<1\right]$

$$\therefore \frac{1}{z^2-4z+3} = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{-3\left[1-\frac{z}{3}\right]} - \frac{1}{-1(1-z)}\right] \quad \left[\because |z|<1 \text{ \& } \left|\frac{z}{3}\right|<1\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3}\left[1-\frac{z}{3}\right]^{-1} + [1-z]^{-1}\right]$$

$$= \frac{-1}{6}\left[1-\frac{z}{3}\right]^{-1} + [1-z]^{-1}$$

$$= \frac{-1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right] + \left[1+z+z^2+z^3+\cdots\right]$$

$$f(z) = \frac{-1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right] + \left(1+z+z^2+z^3+\cdots\right] \quad \left[|z|<1 \text{ \& } \left|\frac{z}{3}\right|<1\right)$$

which is the Laurent series expansion of $f(z)$ for $|z|<1$

iii) $|z|>3$:-

$|z|>3$ i.e, $\left|\frac{z}{3}\right|>1$ i.e $\left|\frac{3}{z}\right|<1$ $\left[\text{and } \left|\frac{1}{z}\right|<1\right]$

$$\frac{1}{z^2-4z+3} = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z\left[1-\frac{3}{z}\right]} - \frac{1}{z\left[1-\frac{1}{z}\right]}\right] \quad \left[\because \left|\frac{1}{z}\right|<1 \text{ \& } \left|\frac{3}{z}\right|<1\right)$$

$$= \frac{1}{2}\left[\frac{1}{z}\left[1-\frac{3}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}\right]$$

$$= \frac{1}{2z}\left[1-\frac{3}{z}\right]^{-1} - \frac{1}{2z}\left[1-\frac{1}{z}\right]^{-1}$$

$$= \frac{1}{2z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right] - \frac{1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\frac{27}{z^4}+\cdots\right] - \frac{1}{2}\left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots\right]$$

$$= \frac{1}{z}\left[\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}\right] + \frac{1}{z^2}\left[\frac{3}{2}-\frac{1}{2}\right] + \frac{1}{z^3}\left[\frac{9}{2}-\frac{1}{2}\right] + \frac{1}{z^4}\left[\frac{27}{2}-\frac{1}{2}\right]+\cdots$$

$$\frac{1}{z^2-4z+3} = \frac{1}{z^2}+\frac{4}{z^3}+\frac{13}{z^4}+\cdots$$

$$\therefore \frac{1}{z^2-4z+3} = \frac{1}{z^2}+\frac{4}{z^3}+\frac{13}{z^4}+\cdots$$

Obtain all the Laurent series of the function $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ about $z_0 = -1$

Sol: Given $f(z) = \frac{7z-2}{(z+1)z(z-2)}$.

This function has three singular points $0, -1, 2$.

By partial fractions

$$f(z) = \frac{7z-2}{(z+1)z(z-2)} = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2} \quad \text{—} (1)$$

Three Laurent series of $f(z)$ can be obtained about $z_0 = -1$ in the regions $0 < |z+1| < 1$, $1 < |z+1| < 3$ & $|z+1| > 3$.

Case (1): For $0 < |z+1| < 1$:-

$z+1 = ?$

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Expand $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ in the region

i) $1<|z|<4$

ii) $|z|<1$

Soln: Given $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$

$$z^2+5z+4 \,)\, z^2-4 \,(\, 1$$
$$\underline{z^2+5z+4}$$
$$-5z-8$$

$$= \frac{z^2-4}{z^2+5z+4}$$

$$= 1 - \frac{5z+8}{z^2+5z+4}$$

$$= 1 - \frac{5z+8}{(z+1)(z+4)}$$

$$\frac{5z+8}{(z+1)(z+4)} = \frac{A}{z+1} + \frac{B}{z+4}$$

$$5z+8 = A(z+4) + B(z+1)$$

Put $z=-4$ Put $z=-1$

$-12 = -3B$ $3 = 3A$

$\boxed{B=4}$ $\boxed{A=1}$

$$= 1 - \left(\frac{1}{z+1} + \frac{4}{z+4}\right)$$

$$\therefore \frac{(z-2)(z+2)}{(z+1)(z+4)} = 1 - \frac{1}{z+1} - \frac{4}{z+4} \quad \text{——} (1)$$

i) $1<|z|<4$:-

$1<|z|<4$

i.e, $\left|\frac{1}{z}\right|<1$ & $\left|\frac{z}{4}\right|<1$

$$(1) \Rightarrow \frac{(z-2)(z+2)}{(z+1)(z+4)} = 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - \frac{1}{z}\left[1+\frac{1}{z}\right]^{-1} - \left[1+\frac{z}{4}\right]^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \left(\frac{z}{4}\right)^3 + \cdots\right]$$

Laurent's Theorem :-

Statement :-

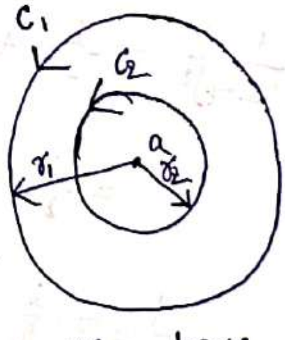
If $f(z)$ is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 & C_2 of radi r_1 and r_2 $(r_1 > r_2)$ respectively having centre at a, then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \text{-----} + a_1(z-a)^{-1} + a_2(z-a)^{-2} + \text{---} \text{------}$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \dfrac{f(w)}{(w-a)^{n+1}}\, dw$, $n = 0, 1, 2, 3, \text{---}$

and $a_{-n} = \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \dfrac{f(w)}{(w-a)^{-n+1}}\, dw$, $n = 1, 2, 3, \text{----}$

Proof :-



Let z be any point in the region 'R'.

By cauchy's integral formular for doubly connected regions, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{w-z}\, dw \quad \longrightarrow (1)$$

Consider the first integral,

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw.$$

Let w lies on C_1. Then $|z-a|<|w-a|$

$$\Rightarrow \left|\frac{z-a}{w-a}\right|<1$$

Now $\frac{1}{w-z}=\frac{1}{w-a+a-z}=\frac{1}{(w-a)-(z-a)}$

$$=\frac{1}{(w-a)\left[1-\frac{z-a}{w-a}\right]}$$

$$=\frac{1}{(w-a)}\left[1-\left(\frac{z-a}{w-a}\right)\right]^{-1}$$

$$=\frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\left(\frac{z-a}{w-a}\right)^3+\cdots\right]$$

Multiplying both sides by $\frac{1}{2\pi i}f(w)$ and integrate term by term w.r.to w along the circle 'C_1'

then

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-a}\,dw+\frac{z-a}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^2}\,dw$$
$$+\frac{(z-a)^2}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^3}\,dw+\cdots$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw=a_0+a_1(z-a)+a_2(z-a)^2+\cdots$$
$$=\sum_{n=0}^{\infty}a_n(z-a)^n \rightarrow (2)$$

where $a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,dw$ for $n=0,1,2,\cdots$

consider the second integral

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw.$$

Let 'w' lies on C_2. Then $|w-a|<|z-a|$.

$$\Rightarrow \left|\frac{w-a}{z-a}\right|<1$$

Now $\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)-(z-a)}$

$$\Rightarrow \frac{1}{w-z} = \frac{-1}{(z-a)\left[1-\frac{w-a}{z-a}\right]}$$

$$\frac{1}{w-z} = -\frac{1}{z-a}\left[1-\left(\frac{w-a}{z-a}\right)\right]^{-1}$$

$$= -\frac{1}{z-a}\left[1+\left(\frac{w-a}{z-a}\right)+\left(\frac{w-a}{z-a}\right)^2+\left(\frac{w-a}{z-a}\right)^3+\cdots\right]$$

Multiplying both sides by $-\frac{1}{2\pi i}f(w)$ and integrating term by term w.r.to w along the circle C_2, we have

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = \frac{1}{z-a}\cdot\frac{1}{2\pi i}\int_{C_2}f(w)\,dw + \frac{1}{(z-a)^2}\cdot\frac{1}{2\pi i}\int_{C_2}(w-a)f(w)\,dw$$

$$+\frac{1}{(z-a)^3}\cdot\frac{1}{2\pi i}\int_{C_2}(w-a)^2 f(w)\,dw+\cdots$$

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = a_{-1}(z-a)^{-1}+a_{-2}(z-a)^{-2}+a_{-3}(z-a)^{-3}+\cdots$$

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw = \sum_{n=1}^{\infty}a_{-n}(z-a)^{-n} \longrightarrow (3)$$

where $a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}\,dw$, $n=1,2,3,\ldots$

substitute ② & ③ in ①, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (x-a)^{-n}$$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

where $$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \quad \text{for } n = 0, 1, 2, 3, \ldots$$

$$a_{-n} = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw \quad \text{for } n = 1, 2, 3, \ldots$$

This is ~~ko~~ known as "Laurent's series".

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① Expand the series

$$f(z) = \frac{1}{z^2-3z+2}$$

(i) $0 < |z| < 1$ ② $|z| > 2$ ③ $0 < |z-1| < 1$.

z^2-3z+2

$= z^2-2z-z+2$

$= z(z-2)-1(z-2)$

$= (z-2)(z-1)$

Sol :- we have

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$A(z-2) + B(z-1) = 1.$

$-A = 1 \Rightarrow A = -1$

$B = 1$

$$f(z) = \frac{1}{z^2-3z+2} = \frac{-1}{z-1} + \frac{1}{z-2}$$

① $0 < |z| < 1$:-

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$= -\frac{1}{-1(1-z)} + \frac{1}{-2(1-\frac{z}{2})}$$

$$= (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \left\{1+z+z^2+z^3+\cdots\right\} - \frac{1}{2}\left\{1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right\}$$

② $|z| > 2$

$\Rightarrow 2 < |z| \Rightarrow \frac{2}{|z|} < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{z(1-\frac{2}{z})} - \frac{1}{z(1-\frac{1}{z})}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left\{1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots\right\} - \frac{1}{z}\left\{1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right\}$$

$$= \left\{\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\cdots\right\} - \left\{\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right\}$$

(3) $0 < |z-1| < 1$

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= \frac{1}{-[1-(z-1)]} - \frac{1}{z-1}$$

$$= -[1-(z-1)]^{-1} - \frac{1}{z-1}$$

$$= -\left\{1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right\} - \frac{1}{z-1}$$

(or)

$z-1 = t \Rightarrow z = 1+t$

$$f(z) = \frac{1}{(t+1)-2} - \frac{1}{(t+1)-1} = \frac{1}{t-1} - \frac{1}{t}$$

Given that $|z-1| < 1 \Rightarrow |t| < 1$

$$f(z) = -\frac{1}{t} - \frac{1}{(1-t)} = -\frac{1}{t} - (1-t)^{-1}$$

$$= -\frac{1}{t} - \left\{1+t+t^2+t^3+\cdots\right\}$$

$$= -\frac{1}{z-1} - \left\{1+(z-1)+(z-1)^2+\cdots\right\}$$

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(1) Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z=1$ as

Laurent's series. Also find the region

of convergence.

Sol :- Given that $f(z) = \frac{e^{2z}}{(z-1)^3}$.

We want Laurent series expansion around $z=1$

put $z=1 \Rightarrow z-1=0$

$\Rightarrow z-1=t \quad \Rightarrow z=1+t$

$$f(z) = \frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+t)}}{(1+t-1)^3} = \frac{e^{2t+2}}{t^3}$$

$$= e^2 \cdot \frac{1}{t^3} \cdot e^{2t}$$

$$= e^2 \cdot \frac{1}{t^3} \left\{ 1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots \right\}$$

$$= e^2 \cdot \frac{1}{t^3} \left\{ \sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot t^n \right. \quad \text{if } t \neq 0$$

$$= e^2 \cdot \sum_{n=0}^{\infty} \frac{2^n}{n!} \frac{t^n}{t^3} \quad \text{if } t \neq 0.$$

$$= e^2 \cdot \sum_{n=0}^{\infty} \frac{2^n}{n!} \left(t^{n-3} \right) \quad \text{if } z-1 \neq 0.$$

$$= e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!} (z-1)^{n-3} \quad \text{if } |z-1| > 0.$$

---x---

(2) Obtain the Taylor's & Laurent's series for the function $f(z) = \frac{1}{(z+1)(z+3)}$ in the following region.

(1) $|z| \leq 1$ (2) $1 < |z| < 3$

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(3) $1 < |z| < 3$ (4) $0 < |z+1| < 2$.

Sol:- Given that $f(z) = \frac{1}{(z+1)(z+3)}$.

(a) $|z| < 1$.

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}.$$

z^2+z+3z

$$= \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$A(z+3)+B(z+1) = 1$

$-2B = 1 \Rightarrow B = -\frac{1}{2}$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$$\therefore f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\cdot\frac{1}{z+1} - \frac{1}{2(z+3)}.$$

$$= \frac{1}{2}\left\{\frac{1}{z+1} - \frac{1}{z+3}\right\}.$$

$$f(z) = \frac{1}{2}\left\{\frac{1}{z+1} - \frac{1}{z+3}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{1+z} - \frac{1}{3\left(1+\frac{z}{3}\right)}\right\} = \frac{1}{2}\left\{(1+z)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right\}$$

$$f(z) = \frac{1}{2}\left\{\left(1-z+z^2-z^3+\cdots\right) - \frac{1}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right)\right\}$$

$$= \left(\frac{1}{2}-\frac{1}{6}\right) + z\left(-\frac{1}{2}+\frac{1}{18}\right) + z^2\left(\frac{1}{2}-\frac{1}{54}\right) + \cdots$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 + \cdots$$

$\frac{3-1}{6}$

$-\frac{26}{18}$

36

$-18+2$

$\frac{4}{36}$

(b) $|z| > 3$

$\Rightarrow 3 < |z| \Rightarrow 1 < \frac{|z|}{3} \Rightarrow \frac{|z|}{3}$

$\Rightarrow \frac{3}{|z|} < 1$

$$f(z) = \frac{1}{2}\left\{ \frac{1}{z+1} - \frac{1}{z+3} \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{z\left(1+\frac{3}{z}\right)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1} \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{z}\left(1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right) \right\}$$

$$= \frac{1}{2}\left\{ \left(\cancel{\frac{1}{z}} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\right) - \left(\cancel{\frac{1}{z}} - \frac{3}{z^2} + \frac{3^2}{z^3} - \frac{3^3}{z^4} + \cdots\right) \right\}$$

$$= \frac{1}{2}\left\{ \frac{2}{z^2} - \frac{8}{z^3} + \frac{26}{z^4} - \cdots \right\}$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots$$

(c) $1 < |z| < 3$.

$$f(z) = \frac{1}{2}\left\{ \frac{1}{z+1} - \frac{1}{z+3} \right\}.$$

$1 < |z| < 3$

$\Rightarrow 1 < |z| \Rightarrow \frac{1}{|z|} < 1$ & $|z| < 3$

$\Rightarrow \frac{|z|}{3} < 1.$

Now

$$f(z) = \frac{1}{2}\left\{ \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{3\left(1+\frac{z}{3}\right)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1} \right\}.$$

$$= \frac{1}{2}\left\{ \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}-\cdots\right) - \frac{1}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right) \right\}$$

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$$f(z) = \frac{1}{2}\left\{\left(\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \cdots\right) - \left(\frac{1}{3} - \frac{2}{3^2} + \frac{2^2}{3^3} - + \cdots\right)\right\}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2^{n+1}} - \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n\right]$$

4 (4) $0 < |z+1| < 2$

Sol :— $f(z) = \frac{1}{2}\left\{\frac{1}{z+1} - \frac{1}{z+3}\right\}$.

$0 < |z+1| < 2 \Rightarrow z+1 = t \Rightarrow z = t-1$

$0 < |z+1| < 2 \Rightarrow 0 < |t| < 2. \Rightarrow |t| < 2 \Rightarrow \frac{|t|}{2} < 1$

$$f(z) = \frac{1}{2}\left\{\frac{1}{t} - \frac{1}{(t-1)+3}\right\} = \frac{1}{2}\left\{\frac{1}{t} - \frac{1}{t+2}\right.$$

$$f(z) = \frac{1}{2}\left\{\frac{1}{t} - \frac{1}{2(1+\frac{t}{2})}\right\}.$$

$$= \frac{1}{2}\left\{\frac{1}{t} - \frac{1}{2(1+\frac{t}{2})}\right\} = \frac{1}{2}\left\{\frac{1}{t} - \frac{1}{2}\left(1+\frac{t}{2}\right)^{-1}\right\}.$$

$$f(z) = \frac{1}{2}\left\{\frac{1}{t} - \frac{1}{2}\left(1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} + \cdots\right)\right\}$$

$$= \frac{1}{2t} - \frac{1}{4}\left(1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} + \cdots\right)\Big\}$$

$$= \frac{1}{2(z+1)} - \frac{1}{4}\left(1 - \frac{z+1}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + \cdots\right)$$

— x ————

LAURENT'S THEOREM:-

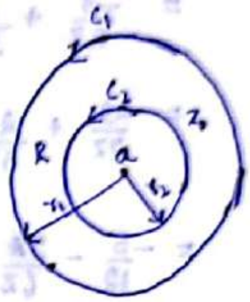
If $f(z)$ is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles 'C_1' and 'C_2' of radii 'r_1' and 'r_2' $(r_1 > r_2)$ respectively having centre at a, then for all 'z' in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots\dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots\dots$$

where $\displaystyle a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw$, $n = 0,1,2,3 \dots\dots$

and $\displaystyle a_{-n} = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-a)^{-n+1}}\,dw$, $n = 1,2,3 \dots\dots$

Proof:- Let 'z' be any point in the region 'R'.



By Cauchy's integral formula for doubly connected regions we have,

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,dw \longrightarrow (1)$$

Consider the first integral, $\displaystyle \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw$

Let 'w' lies on 'C_1'. Then, $|z-a| < |w-a|$

$$\Rightarrow \left|\frac{z-a}{w-a}\right| < 1$$

Now, $$\frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{(w-a)\left[1-\left(\frac{z-a}{w-a}\right)\right]}$$

$$= \frac{1}{w-a}\left[1-\left(\frac{z-a}{w-a}\right)\right]^{-1}$$

$$= \frac{1}{w-a}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\left(\frac{z-a}{w-a}\right)^3+\dots\dots\right]$$

Multiplying both sides by $\frac{1}{2\pi i}f(w)$ and integrate term by term w.r.to 'w' along the circle 'C_1' then

$$\Rightarrow \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \frac{(z-a)^2}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^3}\,dw + \dots\dots\dots$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\,dw = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots\dots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \rightarrow (2) \text{ where } a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega \quad \text{for } n = 0,1,2 \ldots$$

Consider the second integral, of $-\frac{1}{2\pi i}\int_{C_2} \frac{f(\omega)}{\omega - z} d\omega$.

Let 'ω' lies on 'C_2'. Then, $|\omega - a| < |z-a|$

$$\Rightarrow \left|\frac{\omega - a}{z-a}\right| < 1.$$

Now, $\frac{1}{\omega - z} = \frac{1}{\omega - a + a - z} = \frac{1}{(\omega-a)-(z-a)} = \frac{1}{-(z-a)\left[1 - \left(\frac{\omega-a}{z-a}\right)\right]}$

$$= \frac{-1}{z-a}\left[1 - \left(\frac{\omega - a}{z-a}\right)\right]^{-1}$$

$$= \frac{-1}{z-a}\left[1 + \left(\frac{\omega-a}{z-a}\right) + \left(\frac{\omega-a}{z-a}\right)^2 + \left(\frac{\omega-a}{z-a}\right)^3 + \cdots\right]$$

Multiplying both sides by $-\frac{1}{2\pi i} f(\omega)$ and integrating term by term w.r.t 'ω' along the circle 'C_2' we have,

$$-\frac{1}{2\pi i}\int_{C_2} \frac{f(\omega)}{\omega - z} d\omega = \frac{1}{z-a}\cdot\frac{1}{2\pi i}\int_{C_2} f(\omega)\, d\omega + \frac{1}{(z-a)^2}\cdot\frac{1}{2\pi i}\int_{C_2} (\omega-a)\, f(\omega)\, d\omega$$

$$+ \frac{1}{(z-a)^3}\, \frac{1}{2\pi i}\int_{C_2} (\omega-a)^2 f(\omega)\, d\omega + \cdots\cdots$$

$$-\frac{1}{2\pi i}\int_{C_2} \frac{f(\omega)}{\omega - z} d\omega = a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + a_{-3}(z-a)^{-3} + \cdots\cdots$$

$$\frac{-1}{2\pi i}\int_{C_2} \frac{f(\omega)}{\omega - z} d\omega = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \rightarrow (3) \quad \text{where } a_{-n} = \frac{1}{2\pi i}\int_{C_2} \frac{f(\omega)}{(\omega-a)^{-n+1}} d\omega$$

for $n = 1,2,3\ldots$

Substitute (2) & (3) in (1) we have,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$$

$$f(z) = a_0 + a_1(z-a)^1 + a_2(z-a)^2 + \cdots\cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots\cdots$$

where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\omega)}{(\omega-a)^{n+1}} d\omega$ for $n = 0,1,2,3\ldots$

$a_{-n} = \frac{1}{2\pi i}\int_{C_2} \frac{f(\omega)}{(\omega-a)^{-n+1}} d\omega$ for $n = 1,2,3\ldots$

This is known as "LAURENT'S SERIES."

Pro:- Find the Laurent Expansion of $\frac{1}{z^2-4z+3}$ for $1<|z|<3$.

Sol:- Given, $f(z) = \frac{1}{z^2-4z+3}$

z^2-4z+3
$z^2-3z-z+3$
$z(z-3)-1(z-3)$
$(z-3)(z-1)$

Consider, $\frac{1}{z^2-4z+3} = \frac{1}{(z-1)(z-3)} = \frac{A}{z-3} + \frac{B}{z-1}$

$\Rightarrow 1 = A(z-1) + B(z-3)$

Put $z=1 \Rightarrow 1 = -2B \Rightarrow \boxed{B = \frac{-1}{2}}$

Put $z=3 \Rightarrow 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$

$\therefore \frac{1}{(z-1)(z-3)} = \frac{1/2}{z-3} + \frac{(-1/2)}{z-1} = \frac{1}{2(z-3)} - \frac{1}{2(z-1)}$

For $1<|z|<3$

$\Rightarrow 1<|z|$ and $|z|<3$.

$\Rightarrow \frac{1}{|z|}<1$ and $\frac{|z|}{3}<1$.

$\therefore f(z) = \frac{1}{2(-3)[1-z/3]} - \frac{1}{2(z)[1-1/z]} = \frac{-1}{6}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}$

$= \frac{-1}{6}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right) - \frac{1}{2z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right)$

Pro:- Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z=1$ as a Laurent's Series? Also find the region of Convergence?

Sol:- Given, $f(z) = \frac{e^{2z}}{(z-1)^3}$.

Put $z-1 = w \Rightarrow z = w+1$.

Then, $f(w+1) = \frac{e^{2(w+1)}}{w^3} = \frac{e^{2w}.e^2}{w^3}$

$= \frac{e^2}{w^3}\left[e^{2w}\right] = \frac{e^2}{w^3}\left[1+(2w)+\frac{(2w)^2}{2!}+\frac{(2w)^3}{3!}+\frac{(2w)^4}{4!}+\cdots\right]$ if $w \neq 0$

$= e^2\left[\frac{1}{w^3}+\frac{2}{w^2}+\frac{4}{2!\,w}+\frac{8}{3!}+\frac{16w}{4!}+\cdots\right]$ if $|z-1| \neq 0$.

$= e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{4}{2!(z-1)}+\frac{8}{3!}+\frac{16(z-1)}{4!}+\cdots\right]$

if $|z-1|>0$.

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(b) $|z|<1 \Rightarrow \frac{|z|}{3}<1 \Rightarrow \left|\frac{z}{3}\right|<1.$ 8(a)

Sol:- $\therefore f(z) = \frac{1}{2(z-3)} - \frac{1}{2(z-1)}$

$$= \frac{1}{2(-3)(1-\frac{z}{3})} - \frac{1}{2(-1)(1-z)}$$

$$= \frac{-1}{6}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{2}(1-z)^{-1}$$

$$= \frac{-1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\dots\right] + \frac{1}{2}\left[1+z+z^2+z^3+\dots\right]$$

(c) $|z|>3 \Rightarrow \frac{|z|}{3}>1 \Rightarrow \frac{3}{|z|}<1.$

$\therefore f(z) = \frac{1}{2(z-3)} - \frac{1}{2(z-1)}$

$$= \frac{1}{2z(1-\frac{3}{z})} - \frac{1}{2z(1-\frac{1}{z})}$$

$$= \frac{1}{2z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\dots\right] - \frac{1}{2z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right]$$

$$= \frac{1}{2z}\left[1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\dots\right] - \frac{1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right]$$

Problem:- Determine the Laurent's Series Expansion of the function $\frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3<|z+2|<5$.

Sol:- Given, $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z+2}$

By solving $\boxed{A=1}$; $\boxed{B=-1}$ and $\boxed{C=1}$

$$= \frac{1}{z-1} - \frac{1}{z-3} + \frac{1}{z+2}$$

For $3<|z+2|<5$

Put $z+2=\omega \Rightarrow z=\omega-2$

Now, $z-1 = w-2-1 = w-3$

$z-3 = w-2-3 = w-5$

$3 < |z+2| < 5$

$\Rightarrow 3 < |w| < 5$

$\Rightarrow 3 < |w|$ and $|w| < 5$

$\Rightarrow \frac{3}{|w|} < 1$ and $\frac{|w|}{5} < 1$

$\Rightarrow \left|\frac{3}{w}\right| < 1$ and $\left|\frac{w}{5}\right| < 1.$

$\therefore \quad f(z) = \frac{1}{w-3} - \frac{1}{w-5} + \frac{1}{w}$

$$= \frac{1}{w\left(1-\frac{3}{w}\right)} - \frac{1}{-5\left(1-\frac{w}{5}\right)} + \frac{1}{w}$$

$$= \frac{1}{w}\left(1-\frac{3}{w}\right)^{-1} + \frac{1}{5}\left(1-\frac{w}{5}\right)^{-1} + \frac{1}{w}$$

$$= \frac{1}{w}\left[1+\left(\frac{3}{w}\right)+\left(\frac{3}{w}\right)^2+\left(\frac{3}{w}\right)^3+\dots\right] + \frac{1}{5}\left[1+\frac{w}{5}+\left(\frac{w}{5}\right)^2+\left(\frac{w}{5}\right)^3+\dots\right] + \frac{1}{w}.$$

$$= \frac{1}{z+2}\left[1+\frac{3}{z+2}+\frac{9}{(z+2)^2}+\frac{27}{(z+2)^3}+\dots\right] + \frac{1}{5}\left[1+\frac{z+2}{5}+\frac{(z+2)^2}{25}+\frac{(z+2)^3}{125}+\dots\right] + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\sum_{n=0}^{\infty}\left(\frac{3}{z+2}\right)^n + \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z+2}{5}\right)^n + \frac{1}{z+2}.$$

Valid for $\left|\frac{3}{z+2}\right| < 1$ and $\left|\frac{z+2}{5}\right| < 1.$

Pro:- Expand the Laurent Series of $\dfrac{z^2-1}{(z+2)(z+3)}$ for $|z|>3$.

Sol:- Given, $f(z) = \dfrac{z^2-1}{(z+2)(z+3)} \longrightarrow (1)$

Since, $f(z)$ is an improper fraction where degree of Nr. $\geq$ degree of Dr. then,

$$\frac{\text{coeff. of highest degree in Nr.}}{\text{coeff. of highest degree in Dr.}} = \frac{1}{1} = 1.$$

from (1) $\Rightarrow$ $f(z) = \dfrac{z^2-1}{(z+2)(z+3)} = 1 + \dfrac{A}{z+2} + \dfrac{B}{z+3}$.

$$\Rightarrow \quad z^2-1 = (z+2)(z+3) + A(z+3) + B(z+2)$$

Put $z=-2 \Rightarrow 4-1 = (-2+2)(-2+3) + A(-2+3) + B(-2+2) \Rightarrow \boxed{A=3}$

Put $z=-3 \Rightarrow 9-1 = (-3+2)(-3+3) + A(-3+3) + B(-3+2) \Rightarrow -B = 8 \Rightarrow \boxed{B=-8}$

$$\therefore \quad f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

For $|z|>3 \Rightarrow \dfrac{|z|}{3} > 1 \Rightarrow \dfrac{3}{|z|} < 1 \Rightarrow \left|\dfrac{3}{z}\right| < 1$.

Also, $\left|\dfrac{2}{z}\right| < 1$.

$$\therefore \cancel{f(z) = 1 + \frac{3}{2(1+\frac{z}{2})} - \frac{8}{3(1+z/3)} = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}}$$

$$\cancel{= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]}$$

$$\therefore \quad f(z) = 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{z(1+\frac{3}{z})} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right]$$

$$f(z) = 1 + \frac{3}{z}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z}\right)^n.$$

Pro:- Expand $f(z) = \dfrac{1}{z^2-3z+2}$ in the region of (a) $0<|z-1|<1$ (b) $1<|z|<2$.

(B. S. Grewal) (Eg:-)

Sol:- Given, $f(z) = \dfrac{1}{z^2-3z+2}$

Consider, $z^2-3z+2 = z^2-2z-z+2$
$= z(z-2) - 1(z-2)$
$= (z-2)(z-1)$.

$$\therefore \quad f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

(P) $\Rightarrow 1 = A(z-2) + B(z-1)$

Put $z=1 \Rightarrow 1 = -A \Rightarrow \boxed{A=-1}$

Put $z=2 \Rightarrow 1 = B \Rightarrow \boxed{B=1}$

$\therefore \quad f(z) = \dfrac{-1}{z-1} + \dfrac{1}{z-2}.$

$\Rightarrow \quad f(z) = \dfrac{1}{z-2} - \dfrac{1}{z-1} \quad \longrightarrow (1)$

(a) $0<|z-1|<1.$

Put $z-1=t \Rightarrow z = t+1$

Now, $\boxed{z-2 = t+1-2 = t-1}$

from (1) $\Rightarrow f(z) = \dfrac{1}{t-1} - \dfrac{1}{t} = \dfrac{1}{-(1-t)} - \dfrac{1}{t}$

$= -(1-t)^{-1} - \frac{1}{t}$

$= -\frac{1}{t} - (1+t+t^2+t^3+\dots\dots)$ if $t<1$ and $t \neq 0$.

$= \dfrac{-1}{z-1} - \left[1+(z-1)+(z-1)^2+(z-1)^3+\dots\right]$ if $0<|z-1|<1$.

$= \dfrac{-1}{z-1} - \sum_{n=0}^{\infty} (z-1)^n$ if $0<|z-1|<1$.

(b) $1<|z|<2$

$\Rightarrow 1<|z|$ and $|z|<2$

$\Rightarrow \frac{1}{|z|}<1$ and $\frac{|z|}{2}<1$

$\Rightarrow \left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{2}\right|<1.$

$\therefore \quad f(z) = \dfrac{1}{z-2} - \dfrac{1}{z-1} = \dfrac{1}{-2(1-z/2)} - \dfrac{1}{z(1-\frac{1}{z})} = \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$

$= \frac{-1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right]$

$= \frac{-1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\dots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots\right]$

$= \frac{-1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\dots\right] - \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\dots\right].$

Pro:- Obtain Laurent's expansion for $f(z) = \frac{1}{(z+2)(1+z)^2}$ (10)

(a) $|z| < 2$ (b) $|1+z| > 1$.

Sol:- Given, $f(z) = \frac{1}{(z+2)(1+z)^2} = \frac{A}{z+2} + \frac{B}{1+z} + \frac{C}{(1+z)^2}$

$$\rightarrow 1 = A(1+z)^2 + B(z+2)(1+z) + C(z+2)$$

Put $z = -1 \Rightarrow 1 = C(-1+2) \Rightarrow \boxed{C = 1}$

Put $z = -2 \Rightarrow 1 = A(-2+1)^2 \Rightarrow \boxed{A = 1}$

coeff of z^2, $A + B = 0 \Rightarrow B = -A \Rightarrow \boxed{B = -1}$

$$\therefore f(z) = \frac{1}{(z+2)(z+1)^2} = \frac{1}{z+2} + \frac{(-1)}{z+1} + \frac{1}{(z+1)^2}$$

(a) for, $|z| < 2 \Rightarrow \frac{|z|}{2} < 1$.

$$\therefore f(z) = \frac{1}{2(1+\frac{z}{2})} - \frac{1}{1+z} + \frac{1}{(1+z)^2}$$

$$= \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - (1+z)^{-1} + (1+z)^{-2}$$

$$= \frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \dots\right) - \left(1 - z + z^2 - z^3 + \dots\right) + \left(1 - 2z + 3z^2 - \dots\right)$$

(b) For $|1+z| > 1$

Put $1 + z = w \Rightarrow z = w - 1$ Since, $|1+z| > 1 \Rightarrow |w| > 1 \Rightarrow \frac{1}{|w|} < 1$

$$\therefore f(z) = \frac{1}{(z+2)(1+z)^2} = \frac{1}{(w-1+2)(w)^2} = \frac{1}{(w+1)w^2}$$

$$= \frac{1}{w^2(1+w)} = \cancel{\frac{1}{w^2}(1+w)^{-1}} = \frac{1}{w^2 \cdot w(1+\frac{1}{w})}$$

$$\cancel{= \frac{1}{w^2 \cdot w(1+1/w)} = \frac{1}{w^3}\left(1+\frac{1}{w}\right)^{-1}}$$

$$\cancel{\frac{1}{w^3}} \quad \cancel{= \frac{1}{w^2}(1+w)^{-1} = \frac{1}{w^2}\left[w\left(1+\frac{1}{w}\right)\right]^{-1}}$$

$$= \frac{1}{w^3}\left(1+\frac{1}{w}\right)^{-1} = \frac{1}{w^3}\left(1 - \frac{1}{w} + \frac{1}{w^2} - \frac{1}{w^3} + \dots\right)$$

$$= \frac{1}{w^3} - \frac{1}{w^4} + \frac{1}{w^5} - \frac{1}{w^6} + \dots$$

$$= \frac{1}{(1+z)^3} - \frac{1}{(1+z)^4} + \frac{1}{(1+z)^5} - \frac{1}{(1+z)^6} + \dots$$

Prob:- Find the Laurent's expansion of $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in $1<|z+1|<3$. (B.S.Grewal)(Exercise)

Sol:- Given, $f(z) = \frac{7z-2}{(z+1)z(z-2)}$

Put $z+1 = w \Rightarrow z = w-1$

$$\therefore f(z) = \frac{7(w-1)-2}{w(w-1)(w-1-2)} = \frac{7w-9}{w(w-1)(w-3)} = \frac{A}{w} + \frac{B}{w-1} + \frac{C}{w-3}.$$

$$\Rightarrow 7w-9 = A(w-1)(w-3) + Bw(w-3) + Cw(w-1).$$

$w=0 \Rightarrow -9 = A(0-1)(0-3) \Rightarrow 3A = -9 \Rightarrow \boxed{A=-3}$

$w=1 \Rightarrow 7-9 = B(1)(1-3) \Rightarrow -2B = -2 \Rightarrow B = \frac{2}{2} = 1 \Rightarrow \boxed{B=1}$

$w=3 \Rightarrow 21-9 = C(3)(3-1) \Rightarrow 12 = 6C \Rightarrow \boxed{C=2}$

$$\therefore f(z) = -\frac{3}{w} + \frac{1}{w-1} + \frac{2}{w-3}$$

for $1<|z+1|<3 \Rightarrow 1<|w|<3$

$\Rightarrow 1<|w|$ and $|w|<3$

$\Rightarrow \frac{1}{|w|}<1$ and $\frac{|w|}{3}<1$

$\Rightarrow \left|\frac{1}{w}\right|<1$ and $\left|\frac{w}{3}\right|<1$

$$\therefore f(z) = -\frac{3}{w} + \frac{1}{w\left(1-\frac{1}{w}\right)} + \frac{2}{-3\left(1-\frac{w}{3}\right)}$$

$$= -\frac{3}{w} + \frac{1}{w}\left(1-\frac{1}{w}\right)^{-1} - \frac{2}{3}\left(1-\frac{w}{3}\right)^{-1}$$

$$= -\frac{3}{w} + \frac{1}{w}\left(1+\frac{1}{w}+\frac{1}{w^2}+\frac{1}{w^3}+\cdots\right) - \frac{2}{3}\left(1+\frac{w}{3}+\frac{w^2}{9}+\frac{w^3}{27}+\cdots\right)$$

$$= -\frac{3}{w} + \frac{1}{w} + \frac{1}{w^2} + \frac{1}{w^3} + \frac{1}{w^4} + \cdots - \frac{2}{3} - \frac{2w}{9} - \frac{2w^2}{27} - \frac{2w^3}{81} - \cdots$$

$$= -\frac{2}{w} + \frac{1}{w^2} + \frac{1}{w^3} + \frac{1}{w^4} + \cdots - \frac{2}{3} - \frac{2}{9}w - \frac{2}{27}w^2 - \frac{2}{81}w^3 - \cdots$$

$$= \frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \frac{1}{(z+1)^4} + \cdots - \frac{2}{3} - \frac{2}{9}(z+1) - \frac{2}{27}(z+1)^2 - \frac{2}{81}(z+1)^3 - \cdots$$

Neighbourhood of a Point: — A neighbourhood of a Point z_0 is the set of all points z for which $0 < |z - z_0| < \delta$, where 'δ' is very small '+ve' value.

Zero of an analytic function: — A point 'a' is called a zero of an analytic function $f(z)$ if $f(a) = 0$.

Zero of m^{th} order: — If an analytic function $f(z)$ can be expressed in the form

$$f(z) = (z-a)^m \phi(z)$$

where $\phi(z)$ is analytic and $\phi(a) \neq 0$, then $z = a$ is called zero of m^{th} order of the function $f(z)$.

Ex: (a) $f(z) = (z-1)^3$, then $z = 1$ is a zero of Order '3' of $f(z)$.

Singular Points: —

A Singular Point or singularity of a function $f(z)$ is the Point at which the function $f(z)$ ceases to be analytic.

(OR)

A Point $z = a$ at which a function $f(z)$ is not analytic is known as a singular point or singularity of $f(z)$.

Isolated Singularity: — A Point $z = a$ is called an isolated singularity of an analytic function $f(z)$ if

(a) $f(z)$ is not analytic at the point $z = a$.

(b) $f(z)$ is analytic at each point of its neighbourhood

Ex: (a) $f(z) = \dfrac{z+3}{z(z-1)}$ is analytic everywhere except at $z = 0$; $z = 1$.

$\therefore$ $z = 0$ and $z = 1$ are the only two singularities of $f(z)$.

Also there are no other singularities of $f(z)$ in the nbd of $z = 0$ & $z = 1$.

$\therefore$ $z = 0$ and $z = 1$ are the isolated singularities of $f(z)$.

(b) If $f(z) = \dfrac{e^z}{z^2+1}$ then $z = \pm i$ are two isolated singular points of $f(z)$.

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11) **Poles of an Analytic Function:—**

If $z=a$ is an Isolated singular point of an analytic function $f(z)$, then $f(z)$ can be expanded in Laurent's Series about the point $z=a$

i.e., $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n} \longrightarrow (1)$$

The Series of negative integral powers of $(z-a)$ namely $\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$ is known as the "Principal part" of the Laurent's Series of $f(z)$.

" If the Principal part contains a finite number of terms, say m, then the singular Point $z=a$ is called a pole of order m".

If $m=1$, it is called a "Simple pole." (OR) A pole of order one is called a "Simple pole."

Ex:- If $f(z) = \frac{z^2}{(z-1)(z+2)^2}$ then $z=1$ is a simple pole and '$z=-2$' is a pole of order '2'.

Essential Singularity:— If the Principal part of $f(z)$ contains an infinite number of terms i.e., the Series $\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$ contains an infinite number of terms, then the point $z=a$ is called "Essential Singularity of $f(z)$"

(OR)

If the Principal part of $f(z)$ at $z=a$ contains an infinite number of terms, then 'a' is called an "Essential Singularity of $f(z)$."

Ex:- $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$

has infinite number of terms in the negative powers of 'z'.

Hence, $z=0$ is an Essential Singularity.

Pro:- Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$.

Sol:- Given, $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

$\Rightarrow z = 1, -2$ are the poles of $f(z)$

$\therefore$ '$z=1$' is a pole of order '2' and '$z=-2$' is a Simple pole.

(I) Note :-

(1) If we take $a=0$ in Taylor's series expansion, we get

$$f(z) = f(0) + z \cdot f'(0) + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots$$

This is called Maclaurin's series expansion of $f(z)$.

(2) If the function $f(z)$ is not analytic at a point $z=a$, it can't be expanded in taylor series about $z=a$.

— x —

Radius of convergence :-

consider the power series $\sum_{n=0}^{\infty} a_n z^n$.

The Radius of convergence R of power series is defined by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

If $R=\infty$, the function is analytic every where.

— x —

① Expand the following functions as taylor series

ⓐ $f(z) = e^{z}$ ⓑ $f(z) = \cos z$ ⓒ $f(z) = \sin z$.

ⓐ Sol :- Taylor's series is

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots\cdots$$

about $a = 0$

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots\cdots$$

$f(z) = e^{z} \Rightarrow f(0) = e^{0} = 1$

$f'(z) = e^{z} \Rightarrow f'(0) = e^{0} = 1$

$f''(z) = e^{z} \Rightarrow f''(0) = e^{0} = 1.$

$$\therefore e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\cdots$$

Radius of convergence :- $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) z^n.$

$e^{z} = \frac{z^n}{n!}$ Here $a_n = \frac{1}{n!}$ & $a_{n+1} = \frac{1}{(n+1)!}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$$R = \lim_{n\to\infty} \left|\frac{1}{n!} \times \frac{(n+1)!}{1}\right| = \lim_{n\to\infty} \left|\frac{n!(n+1)}{n!}\right|$$

$$= \lim_{n\to\infty} (n+1) = \infty.$$

$\therefore R = \infty.$

$\therefore f(z) = e^{z}$ is analytic every where.

———x———

(2) Expand the following functions as Taylors series :-

(1) $\log z$ at $z=1$ (2) $\sin z$ about $z=\frac{\pi}{2}$

(3) e^z about $z=1$ (4) $\cos hz$ about $z=\pi i$

Sol :- $f(z) = \log z$

we know Taylor's series about $z=a$ is

$$f(z) = f(a) + (z-a) f'(a) + (z-a)^2 \cdot \frac{f''(a)}{2!} + \cdots$$

$f(z) = \log z$ $\quad$ $f(1) = \log 1 = 0$

$f'(z) = \frac{1}{z}$ $\quad$ $f'(1) = 1$

$f''(z) = -\frac{1}{z^2}$ $\quad$ $f''(1) = -1$

$f'''(z) = \frac{2}{z^3}$ $\quad$ $f'''(1) = 2$

$$\log z = 0 + (z-1)\cdot 1 + (z-1)^2 \cdot \frac{(-1)}{2!} + \frac{(z-1)^3}{3!} \cdot 2 + \cdots$$

$$\Rightarrow \log z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{2}{3!}(z-1)^3 + \cdots$$

$$\log z = (z-1) - \frac{(z-1)^2}{2!} + \frac{2}{2.3}(z-1)^3 + \cdots$$

$$\log z = (z-1) - \frac{(z-1)^2}{2!} + \frac{(z-3)^3}{3} + \cdots$$

(3) e^z about $z=1$

Sol :- we want taylor's series expansion around $z=1$.

put $z = 1 \Rightarrow z-1=0$

$\Rightarrow z-1 = t \Rightarrow t=0$

$\therefore z = 1+t.$

$\therefore$ we know expansion of e^z

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

$\therefore$ Here we have $z = 1+t$

$$\therefore e^z = \cancel{1+(1+t)} + \cancel{\frac{(1+t)^2}{2!}}$$

$$e^z = e^{1+t} = e \cdot e^t = e\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right]$$

$$\therefore e^z = e\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots\right].$$

— x —

(3) Obtain the expansion of $\frac{1}{(z-1)(z-3)}$ in Taylor's series in powers of $(z-4)$ and determine the region of convergence.

Sol :-

Let $f(z) = \frac{1}{(z-1)(z-3)}$.

The singular points are $(z-1)(z-3)=0 \Rightarrow z=1$ & $z=3$

i.e, $f(z)$ is not analytic at $z=1$ & $z=3$.

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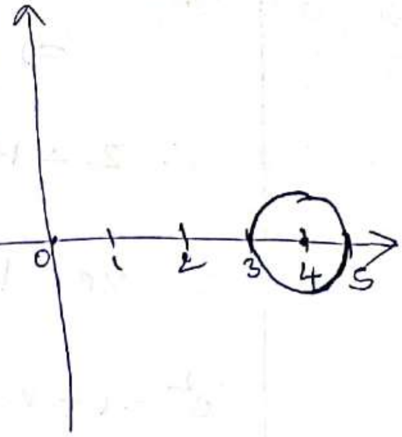
If the center of the circle is at $z=4$, then the distances of the singularities $z=1$ & $z=3$ from the centers are 3 & 1.



Then within the circle $|z-4|=1$ the given function $f(z)$ is analytic hence it can be expanded in a taylor's series within the circle $|z-4|=1$, which is the circle of convergence.

let $(z-4)=t$ & $z=t+4$

$$f(z)=\frac{1}{(z-1)(z-3)}=\frac{1}{(t+3)(t+1)}$$

$$\frac{1}{(t+3)(t+1)}=\frac{A}{t+3}+\frac{B}{t+1}=-\frac{1}{2}\cdot\frac{1}{t+3}+\frac{1}{2}\cdot\frac{1}{t+1}$$

$A(t+1)+B(t+3)=1$

$\Rightarrow$ if $t=-1 \Rightarrow 2B=1$ & $B=\frac{1}{2}$

if $t=-3 \Rightarrow -2A=1$ & $A=-\frac{1}{2}$.

$$f(z)=\frac{1}{(z-1)(z-3)}=\frac{1}{2}\left\{\frac{1}{t+1}-\frac{1}{t+3}\right\}$$

$$=\frac{1}{2}\left\{(t+1)^{-1}-\frac{1}{3(1+\frac{t}{3})}\right\}$$

$$f(z) = \frac{1}{2}\left\{(1+t)^{-1} - \frac{1}{3}\left(1+\frac{t}{3}\right)^{-1}\right\}.$$

$$= \frac{1}{2}\left\{\left\{1-t+t^2-t^3+\cdots\right\} - \frac{1}{3}\left\{1-\frac{t}{3}+\frac{t^2}{9}-\frac{t^3}{3^3}+\cdots\right\}\right\}$$

$$= \frac{1}{2}\left(1-t+t^2-t^3+\cdots\right) - \frac{1}{6}\left(1-\frac{t}{3}+\frac{t^2}{9}-\frac{t^3}{3^3}+\cdots\right)$$

$$\left[(1+x)^{-1} = 1-x+x^2-x^3+\cdots\right]$$

$$f(z) = \left(\frac{1}{2}-\frac{1}{6}\right) + t\left(\frac{1}{18}-\frac{1}{2}\right) + t^2\left(\frac{1}{2}-\frac{1}{54}\right) + \cdots$$

$$f(z) = \left(\frac{1}{2}-\frac{1}{3}\right) + (z-4)\left(\frac{1}{18}-\frac{1}{2}\right) + (z-4)^2\left(\frac{1}{2}-\frac{1}{54}\right) + \cdots$$

$$f(z) = \frac{1}{3} - \frac{4}{9}(z-4) + \frac{13}{27}(z-4)^2 + \cdots$$

— x ——

(3) Expand $f(z) = \frac{z-1}{z+1}$ in taylor's series about the point (i) $z=0$ (ii) $z=1$.

Sol:— $f(z) = \frac{z-1}{z+1}$

$\therefore f(z) = \frac{z-1}{z+1} = 1 - \frac{2}{z+1}$

$f(z) = 1 - 2(1+z)^{-1}$

(i) about $z=0$

$f(z) = 1 - 2\left\{1 - z + z^2 - z^3 + \cdots\right\}$

$= -1 + 2z - 2z^2 + 2z^3 - \cdots$

$= -1 + 2\left(z - z^2 + z^3 - \cdots\right)$

if $|z|<1$

$$z+1\overline{)\,z-1\,}\left(1\right.$$
$$\underline{-z+1}$$
$$-2$$

(or)

$\frac{z-1}{z+1} = \frac{z+1-1-1}{z+1}$

$= \frac{(z+1)-(2)}{z+1}$

$= 1 - \frac{2}{z+1}$

(2) **expansion about $z=1$**

$z-1=0$

$\Rightarrow z-1=t \quad \& \quad z=1+t$

$$f(z)=\frac{z-1}{z+1} = 1-2(1+z)^{-1}$$

$$= 1-2\,\frac{1}{1+z} = 1-2\,\frac{1}{1+(1+t)}$$

$$\therefore f(z) = 1-2\,\frac{1}{(2+t)} = 1-2\,\frac{1}{2(1+\frac{t}{2})}$$

$$= 1-\left(1+\frac{t}{2}\right)^{-1}$$

$$= 1-\left\{1-\frac{t}{2}+\frac{t^2}{2^2}-\frac{t^3}{2^3}+----\right\}$$

$$= \frac{t}{2}-\frac{t^2}{2^2}+\frac{t^3}{2^3}+----$$

$$= \frac{z-1}{2}-\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+----$$

— × —

(2) **expansion about $z=1$**

$z-1=0$

$\Rightarrow z-1=t$ & $z=1+t$

$$f(z)=\frac{z-1}{z+1}=1-2(1+z)^{-1}$$

$$=1-2\frac{1}{1+z}=1-2\frac{1}{1+(1+t)}$$

$$\therefore f(z)=1-2\frac{1}{(2+t)}=1-2\frac{1}{2(1+\frac{t}{2})}$$

$$=1-(1+\frac{t}{2})^{-1}$$

$$=1-\left\{1-\frac{t}{2}+\frac{t^2}{2^2}-\frac{t^3}{2^3}+\cdots\right\}$$

$$=\frac{t}{2}-\frac{t^2}{2^2}+\frac{t^3}{2^3}+\cdots$$

$$=\frac{z-1}{2}-\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots$$

— X —

Remarks :- 1) Taylor's theorem tells us that it is not necessary that $f(z)$ be analytic on the boundary of the circle c, what is required is that $f(z)$ should be analytic inside the boundary.

(2) Since $f(z)$ is analytic at all the points inside a circle c, then it is called region of convergence of Taylor's series of $f(z)$.

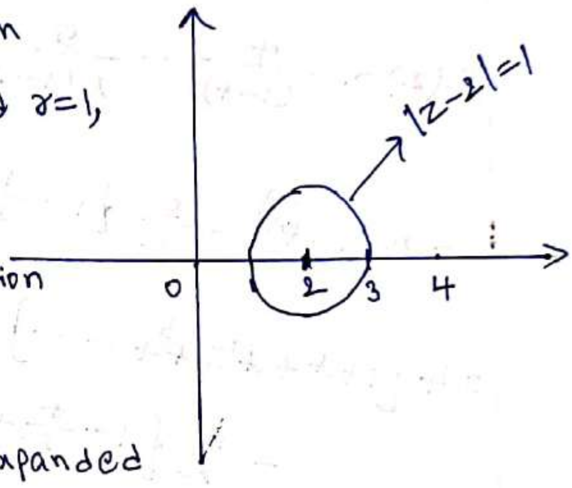
(1) Find Taylor's expansion of $f(z) = \frac{z+1}{(z-3)(z-4)}$ about the point $z=2$. Determine the region of convergence.

Sol:- Given that $f(z) = \frac{z+1}{(z-3)(z-4)}$.

The singular points are $(z-3)(z-4)=0$

$\Rightarrow z=3$ & $z=4$.

If a circle is drawn with centre $z=2$ and $r=1$, then within the circle $|z-2|=1$, the given function $f(z)$ is analytic.



Hence it can be expanded in taylor's series within the circle $|z-2|=1$, which is circle of convergence.

$$f(z) = \frac{z+1}{(z-3)(z-4)}.$$

$$\frac{z+1}{(z-3)(z-4)} = \frac{A}{z-3} + \frac{B}{z-4}$$

$$= -\frac{4}{z-3} + \frac{5}{z-4}$$

$$\therefore f(z) = \frac{z+1}{(z-3)(z-4)} = -\frac{4}{z-3} + \frac{5}{z-4}$$

We have to find taylor's expansion about $z=2$ $\Rightarrow z-2=0$ let $z-2=t \Rightarrow z=t+2$

$$f(z) = -\frac{4}{z-3} + \frac{5}{z-4}$$

$$= -\frac{4}{(t+2)-3} + \frac{5}{(t+2)-4}$$

$$= -\frac{4}{t-1} + \frac{5}{t-2}$$

$$f(z) = -\frac{4}{(-)(1-t)} + \frac{5}{-2\left(1-\frac{t}{2}\right)}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$f(z) = \frac{4}{(1-t)} - \frac{5}{2\left(1-\frac{t}{2}\right)}$$

$$= 4(1-t)^{-1} - \frac{5}{2}\left\{1-\frac{t}{2}\right\}^{-1}$$

$$= 4\left\{1 + t + t^2 + t^3 + \cdots\right\} - \frac{5}{2}\left\{1 + \frac{t}{2} + \frac{t^2}{2^2} + \frac{t^3}{2^3} + \cdots\right\}$$

$$= \left(4 - \frac{5}{2}\right) + t\left(4 - \frac{5}{4}\right) + t^2\left(4 - \frac{5}{8}\right) + \cdots$$

$$= \frac{3}{2} + \frac{11}{4}t + \frac{27}{8}t^2 + \cdots$$

$$= \frac{3}{2} + \frac{11}{4}(z-2) + \frac{27}{8}(z-2)^2 + \cdots$$

— × —

(2) Find Taylor's series expansion for the function $f(z) = \frac{1}{(1+z)^2}$ with center at $-i$.

Sol :- By Taylor's theorem,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots$$

put $a = -i$

$$f(z) = f(-i) + (z+i) f'(-i) + \frac{(z+i)^2}{2!} f''(-i) + \dots + \frac{(z+i)^n}{n!} f^n(-i) + \dots$$

Here $f(z) = \frac{1}{(1+z)^2}$ $\Rightarrow f(-i) = \frac{1}{(1-i)^2} = \frac{1}{1+i^2-2i} = -\frac{1}{2i}$

$$\Rightarrow f(-i) = -\frac{1}{2i} \times \frac{i}{i} = -\frac{i}{-2} = \frac{i}{2}$$

$(\because i^2 = -1)$.

$$f'(z) = -\frac{2}{(1+z)^3}$$

$$f''(z) = \frac{2.3}{(1+z)^4} \quad , \quad f'''(z) = -\frac{2.3.4}{(1+z)^5}$$

|||ly

$$\therefore f^n(z) = (-1)^n \frac{(n+1)!}{(1+z)^{n+2}}$$

$$\& \ f^n(-i) = (-1)^n \frac{(n+1)!}{(1-i)^{n+2}}$$

$$\therefore \frac{1}{(1+z)^2} = \frac{i}{2} + \sum_{n=1}^{\infty} \frac{(z+i)^n}{n!} \frac{(-1)^n (n+1)!}{(1-i)^{n+2}}$$

$$= \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{(z+i)^n (n+1)}{(1-i)^{n+2}}$$

$$\frac{1}{(1+z)^2} = \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n (n+1) . \frac{(z+i)^n}{(1-i)^n (1-i)^2} \quad (\because (1-i)^2 = -2i)$$

$$= \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n (n+1) \frac{(z+i)^n}{(1-i)^n (-2i)} \quad \left(\because \frac{1}{-2i} = \frac{i}{2}, \ -\frac{1}{2i} = \frac{i}{2}\right)$$

$$\frac{1}{(1+z)^2} = \frac{i}{2} + \frac{i}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) \frac{(z+i)^n}{(1-i)^n}$$

— x —

③ obtain the taylor series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the region $|z|<2$.

sol :- let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ $(|z|<2)$. (Improper fraction)

$$z^2+5z+6 \,\big)\, z^2-1 \,\big(\, 1$$
$$\underline{(-)\; z^2+5z+6}$$
$$-5z-7.$$

$\frac{7}{2} = 3\frac{1}{2}$

$$\frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{(5z+7)}{(z+2)(z+3)}.$$

$$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3} = \frac{A(z+3)+B(z+2)}{(z+2)(z+3)}$$

$$5z+7 = A(z+3)+B(z+2).$$

$z=-3 \Rightarrow -B = -8 \Rightarrow B = 8.$

$z=-2 \Rightarrow A = -3 \Rightarrow A=-3.$

$$\frac{5z+7}{(z+2)(z+3)} = -\frac{3}{z+2} + \frac{8}{z+3}$$

$$\therefore f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 - \left[\frac{-3}{z+2} + \frac{8}{z+3}\right] = 1 - \left[\frac{-3}{2\left(1+\frac{z}{2}\right)} + \frac{8}{3\left(1+\frac{z}{3}\right)}\right]$$

$[\because |z|<2]$

$$= 1 - \left[-\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} + \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}.$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n - \frac{8}{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \cdot z^n$$

which is the required taylor's series.

Contour Integration

we know the Laurent's series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ → principal part

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}} dw \quad [n=0,1,2,3,----]$

$b_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{-n+1}} dw \quad [n=1,2,3,----]$

Residue :-

In Laurent's expansion of any function $f(z)$ about the coefficient of $(z-z_0)^{-1}$ $[\because$ i.e, $b_1]$ is defined as the isolated singularity $z=z_0$. residue of $f(z)$ at $z=z_0$.

From Laurent series, $b_1 = \frac{1}{2\pi i}\int_C f(z)dz$.

i.e, $\int_C f(z)dz = 2\pi i \cdot b_1 = 2\pi i\,[\text{Res. of } f(z) \text{ at } z=z_0]$

$= 2\pi i\,[\text{Res } f(z)]_{z=z_0}$

Example :- Find the residue of $z^{-4} e^{iaz}$ at $z=0$.

$f(z) = \frac{1}{z^4}\left\{1+(iaz)+\frac{(iaz)^2}{2!}+\frac{(iaz)^3}{3!}+----\right\}$

$= \frac{1}{z^4}+\frac{ia}{z^3}+\frac{(ia)^2}{2!\,z^2}+\frac{(ia)^3}{z\,3!}+---\quad [\because e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+---]$

Here $b_1 = \frac{a^3 i^3}{3!} = \frac{-ia^3}{6}$

x

Residue at simple pole :-

If the principal part of Laurent's series expansion of any function $f(z)$ contains only single term $b_1(z-z_0)^{-1}$ then the singularity at z_0 is known as single pole (or) pole of order one.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n(z-z_0)^n + b_1(z-z_0)^{-1}$$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0}$$

$$(z-z_0)f(z) = (z-z_0)\sum_{n=0}^{\infty} a_n(z-z_0)^n + b_1$$

$$\underset{z\to z_0}{Lt}\{(z-z_0)f(z)\} = \underset{z\to z_0}{Lt}\{\underbrace{(z-z_0)\sum_{n=0}^{\infty} a_n(z-z_0)^n}_{\text{is zero}} + b_1\}$$

$$\therefore \boxed{b_1 = \lim_{z\to z_0}(z-z_0)f(z).}$$

Find residue at its poles :-

1. (a) $\dfrac{3z+1}{(z+1)(2z-1)}$ (b) $f(z) = \dfrac{z^2}{z^2+4}$.

(a) Sol :- Given that $f(z) = \dfrac{3z+1}{(z+1)(2z-1)}$

The poles of $f(z)$ are $(z+1)(2z-1)=0$

$\Rightarrow z=-1$ & $z=\frac{1}{2}$

$f(z)$ has two simple poles

at $z=-1$ & $z=\frac{1}{2}$

Residue at $z=-1$:-

$$[\text{Res } f(z)]_{z=-1} = \underset{z\to -1}{Lt} (z-(-1))\, f(z)$$

$$= \underset{z\to -1}{Lt} (z+1)\cdot \frac{3z+1}{(z+1)(2z-1)} = \frac{-3+1}{-3} = \frac{2}{3}$$

Residue at $z=\frac{1}{2}$:-

$$[\text{Res } f(z)]_{z=\frac{1}{2}} = \underset{z\to \frac{1}{2}}{Lt} \left(z-\frac{1}{2}\right) \frac{3z+1}{(z+1)(2z-1)}$$

$$= \underset{z\to \frac{1}{2}}{Lt} \frac{(2z-1)}{2} \cdot \frac{(3z+1)}{(z+1)(2z-1)}$$

$$= \frac{1}{2} \cdot \underset{z\to \frac{1}{2}}{Lt} \frac{3z+1}{z+1} = \frac{1}{2} \cdot \frac{\frac{3}{2}+1}{\frac{1}{2}+1}$$

$$= \frac{1}{2}\cdot\frac{5}{3} = \frac{5}{6}$$

(b) Given that $f(z) = \frac{z^2}{z^2+4}$

Sol:- $f(z) = \frac{z^2}{z^2+4} = \frac{z^2}{(z+2i)(z-2i)}$

$\Rightarrow z=-2i$ & $2i$ both are simple poles.

Res at $z=2i$:-

$$[\text{Res } f(z)]_{z=2i} = \underset{z\to 2i}{Lt} (z-2i)\cdot \frac{z^2}{(z+2i)(z-2i)}$$

$$= \underset{z\to 2i}{Lt} \frac{z^2}{z+2i} = \frac{(2i)^2}{4i} = \frac{4(-1)}{4i}$$

$$= -\frac{1}{i} = i$$

|||ly $[\text{Res } f(z)]_{z=-2i} = -i$

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Residue at a pole of order m :-

If the Laurent series with negative powers has a finite number of terms say m, then $z=z_0$ is called a pole order m.

$$\left[\operatorname{Res} f(z)\right]_{z=z_0} = \underset{z\to z_0}{Lt} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z)\right]$$

1. Find the poles and residues at each pole of $\tanh z$.

Sol :-

$\sinh z = \frac{e^z - e^{-z}}{2}$

$\cosh z = \frac{e^z + e^{-z}}{2}$

$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

Let $f(z) = \tanh z = \frac{\sinh z}{\cosh z}$

$$f(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^z - \frac{1}{e^z}}{e^z + \frac{1}{e^z}} = \frac{e^{2z}-1}{e^{2z}+1}$$

Poles of $f(z)$ are given by $e^{2z}+1=0$

i.e, $e^{2z}+1 = (e^z+i)(e^z-i)$

$\Rightarrow e^z = i$ & $-i$ $\Rightarrow e^z = e^{i\frac{\pi}{2}}$ & $e^{-i\frac{\pi}{2}}$

$\Rightarrow z = \frac{\pi}{2}i$ & $-\frac{\pi}{2}i$

$\therefore z = \pm i\frac{\pi}{2}$ are simple poles of $f(z)$.

$$[\text{Res } f(z)]_{z=i\frac{\pi}{2}} = \underset{z\to i\frac{\pi}{2}}{Lt} (z-\alpha)f(z)$$

$$= \underset{z\to i\frac{\pi}{2}}{Lt} \left(z-i\frac{\pi}{2}\right).$$

Now let $f(z) = \frac{e^{2z}-1}{e^{2z}+1} = \frac{\phi(z)}{\psi(z)}$

Res of $f(z)$ at $z = \frac{\pi}{2}i$ is $\frac{\phi(z)}{\psi'(z)}$

$$= \frac{\phi(\frac{\pi}{2}i)}{\psi'(\frac{\pi}{2}i)} = \left[\frac{e^{2z}-1}{\frac{d}{dz}(e^{2z}+1)}\right]_{z=\frac{\pi}{2}i}$$

$$= \left[\frac{e^{2z}-1}{2e^{2z}}\right]_{z=\frac{\pi}{2}i}$$

$$= \frac{1}{2}\left\{1-e^{-2z}\right\} = \frac{1}{2}\left(1-e^{-2\cdot\frac{\pi}{2}i}\right)$$

$$= \frac{1}{2}\left(1-e^{-\pi i}\right)$$

$$= \frac{1}{2}\left(1-(-1)\right) = 1$$

$e^{-\pi i}$

$= \cos(-\pi)+i\sin(-\pi)$

$= -1$

$\left[\text{Res } f(z)\right]_{z=-i\frac{\pi}{2}}$ is

$$= \left[\frac{e^{2z}-1}{2e^{2z}}\right]_{z=-i\frac{\pi}{2}} = \frac{1}{2}\left[1-e^{-2z}\right]_{z=-i\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(1-e^{-2\left(-i\frac{\pi}{2}\right)}\right) = \frac{1}{2}\left(1-e^{i\pi}\right)$$

$$= \frac{1}{2}\left(1-(-1)\right) = 1$$

— x —

(1) Find the residue of $\frac{ze^z}{(z-1)^3}$ at its pole.

Sol:- Let $f(z) = \frac{ze^z}{(z-1)^3}$.

poles of $f(z)$ are obtained by putting the denominator equal to zero.

$\therefore$ $z=1$ is a pole of $f(z)$ of order 3.

we know that if $f(z)$ has a pole of order m at $z=a$ then (4)

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \frac{1}{(m-1)!} \underset{z\to a}{Lt} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m \cdot f(z)\right],$$ here $a=1$ & $m=3$.

$$= \frac{1}{(3-1)!} \underset{z\to 1}{Lt} \frac{d^2}{dz^2}\left[(z-1)^3 \cdot \frac{ze^z}{(z-1)^3}\right]$$

$$= \frac{1}{2!} \underset{z\to 1}{Lt} \frac{d^2}{dz^2}\left(ze^z\right).$$

$$= \frac{1}{2} \underset{z\to 1}{Lt} \frac{d}{dz}\left(ze^z + e^z\right)$$

$$= \frac{1}{2} \underset{z\to 1}{Lt} \left(ze^z + e^z + e^z\right)$$

$$= \frac{1}{2}(3e) = \frac{3}{2}e.$$

① Find the residue of

$$\frac{z^2-2z}{(z+1)^2(z^2+1)}.$$

Sol:-

Let $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+1)} = \frac{z^2-2z}{(z+1)^2(z^2-i^2)}$.

$$= \frac{z^2-2z}{(z+1)^2(z-i)(z+i)}.$$

∴ Poles of $f(z)$ are -1, i & $-i$

⇒ -1 is a pole of order 2.

The poles $\pm i$ are of order one.

$$[\text{Res } f(z)]_{z=-1} = \frac{1}{(m-1)!} \underset{z\to -1}{\text{Lt}} \left[\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)\right].$$

$$= \frac{1}{(2-1)!} \underset{z\to -1}{\text{Lt}} \left[\frac{d}{dz} (z+1)^2 \cdot \frac{z^2-2z}{(z+1)^2 \cdot (z^2+1)}\right]$$

$$= \underset{z\to -1}{\text{Lt}} \left[\frac{d}{dz}\left(\frac{z^2-2z}{z^2+1}\right)\right]$$

$$= \lim_{z\to -1} \left[\frac{(z^2+1)\cdot(2z-2) - (z^2-2z)(2z)}{(z^2+1)^2}\right]$$

$$= \frac{(2)(-4) - (+3)(-2)}{4} = -\frac{2}{4} = -\frac{1}{2}$$

$$[\text{Res } f(z)]_{z=i} = \underset{z\to i}{\text{Lt}} [(z-i) f(z)] = \underset{z\to i}{\text{Lt}} \left[\frac{z^2-2z}{(z+1)^2 (z+i)}\right]$$

$$= \frac{-1-2i}{(1+i)^2(2i)} = \frac{-(1+2i)}{(1-1+2i)(2i)}$$

$$= -\frac{(1+2i)}{4i^2} = \frac{1+2i}{4} \to ①$$

Replacing $i = -i$ in ①

$$[\text{Res } f(z)]_{z=-i} = \frac{1-2i}{4}.$$

MODULE - V

Conformal Mapping & Bilinear transformation

Conformal mapping :-

Problems:-

1) Under the transformation $w = \frac{1}{z}$, find the image of the circle $|z-2i| = 2$

Sol^n: Given transformation $w = \frac{1}{z}$

Let $z = x+iy$ and $w = u+iv$

$$w = \frac{1}{z}$$

$$u+iv = \frac{1}{x+iy}$$

$$= \frac{x-iy}{(x+iy)(x-iy)}$$

$$\therefore u+iv = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

i.e, $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

$$\therefore x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2} \quad \text{---} (1)$$

Given $|z-2i| = 2$ is a circle in z-plane passing through origin.

$$|(x+iy)-2i| = 2$$

$$|x+i(y-2)| = 2$$

$$x^2+(y-2)^2 = 4$$

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$$x^2+y^2-4y+4=0$$

$$x^2+y^2-4y=0$$

Now sub eqn the values of x and y from eqn ①, we get

$$\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}+4\frac{v}{(u^2+v^2)}=0$$

$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2}=0$$

$$(u^2+v^2)+4v(u^2+v^2)=0$$

$$(u^2+v^2)(1+4v)=0$$

$$1+4v=0$$

i.e., $v=-\frac{1}{4}$ which is a straight line in the w-plane.

2) Find the image of the infinite strip $0<y<1/2$ under the transformation $w=1/z$

Soln: Given transformation $w=1/z$

$$z=1/w$$

Let $z=x+iy$ & $w=u+iv$

$$\therefore\ z=\frac{1}{w}$$

$$x+iy=\frac{1}{u+iv}$$

$$x+iy=\frac{u-iv}{(u+iv)(u-iv)}$$

$$x+iy=\frac{u-iv}{u^2+v^2}=\frac{u}{u^2+v^2}+i\left(\frac{-v}{u^2+v^2}\right)$$

$$x=\frac{u}{u^2+v^2}\quad,\quad y=\frac{-v}{u^2+v^2}$$

Given infinite strip $0<y<1/2$

Suppose $y=0 \Rightarrow \dfrac{-v}{u^2+v^2}=0$

$$\boxed{v=0}$$

and $y=1/2 \Rightarrow \dfrac{-v}{u^2+v^2}=1/2$

$$-2v=u^2+v^2$$

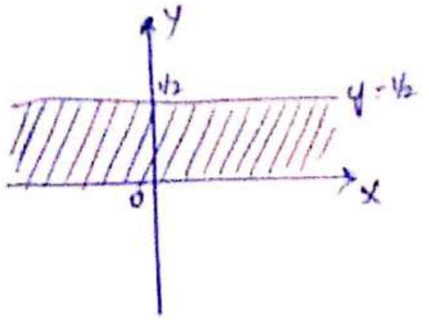
$$\boxed{u^2+v^2+2v=0}$$

$$\Rightarrow u^2+(v+1)^2=1$$

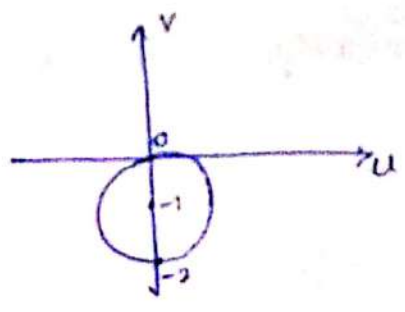
which is a circle of centre $(0,-1)$ and radius 1.

Hence under the transformation $\omega=\frac{1}{z}$, the straight line $y=0$ is transformed to line $v=0$ and the line $y=1/2$ is transformed to the circle $\{u^2+v^2+2v\}$ $u^2+(v+1)^2=1$.

Hence the infinite strip $0<y<1/2$ in z-plane is mapped into the region b/w the line $v=0$ and the circle $u^2+(v+1)^2=1$ in ω-plane under the transformation $\omega=1/z$



z-plane



ω-plane

3) Show that the image of the hyperbola $x^2-y^2=1$ under the transformation $\omega = \frac{1}{z}$ is the laminiscate $\rho^2 = \cos 2\phi$

Sol^n: Given transformation $\omega = \frac{1}{z}$

Let $z = re^{i\theta}$ and $\omega = \rho e^{i\phi}$

$$\omega = \frac{1}{z}$$

$$\rho e^{i\phi} = \frac{1}{re^{i\theta}}$$

$$\rho e^{i\phi} = \left(\frac{1}{r}\right)e^{i(-\theta)}$$

i.e., $\rho = \frac{1}{r}$, $\phi = -\theta$

Given hyperbola, $x^2-y^2=1$

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 1 \quad [\because x = r\cos\theta,\ y = r\sin\theta]$$

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$

$$r^2(\cos^2\theta - \sin^2\theta) = 1$$

$$\cos^2\theta - \sin^2\theta = \frac{1}{r^2}$$

$$\cos 2\theta = \frac{1}{r^2}$$

$$\Rightarrow \cos 2(-\phi) = \rho^2 \quad [\because \rho = \tfrac{1}{r}\ \&\ \theta = -\phi]$$

$$\cos 2\phi = \rho^2$$

$$\therefore \rho^2 = \cos 2\phi$$

$\therefore$ The hyperbola $x^2-y^2=1$ in the z-plane is mapped into laminiscate $\rho^2 = \cos 2\phi$ in the ω-plane.

4) Show that the transformation $w = \frac{2z+3}{z-4}$ changes the circle $x^2+y^2-4x=0$ into the straight line $4u+3=0$

Sol^n: Given transformation is $w = \frac{2z+3}{z-4}$

Now solving z, $w(z-4) = 2z+3$

$$wz - 2z = 3+4w$$

$$z(w-2) = 3+4w$$

$$z = \frac{3+4w}{w-2} \quad \text{——} ①$$

Hence, $\bar{z} = \frac{3+4\bar{w}}{\bar{w}-2}$

The eq^n of the circle $x^2+y^2-4x=0$ is written as $z\bar{z}-2(z+\bar{z})=0$ —— ③

$[\because (x+iy)(x-iy)-2((x+iy)+(x-iy))=0$

$x^2+y^2-2(2x)=0]$

Thus the image of the circle is

$$\left(\frac{3+4w}{w-2}\right)\left(\frac{3+4\bar{w}}{\bar{w}-2}\right) - 2\left(\frac{3+4w}{w-2} + \frac{3+4\bar{w}}{\bar{w}-2}\right) = 0$$ [$\because$ sub eq^n ① & ② in eq^n ③]

$$\Rightarrow \frac{9+12\bar{w}+12w+16w\bar{w}}{(w-2)(\bar{w}-2)} - 2\left(\frac{3\bar{w}+4w\bar{w}-\cancel{6}-8w+3w+4w\bar{w}-6-8\bar{w}}{(w-2)(\bar{w}-2)}\right) = 0$$

$$\Rightarrow 9+12\bar{w}+12w+16\cancel{w}\bar{w} - 6\bar{w} - 8w\bar{w} + 12 + 16w - 6w - 8w\bar{w} + 12 + 16\bar{w} = 0$$

$$\Rightarrow 22\bar{w} + 22w + 33 = 0$$

$$2\bar{w} + 2w + 3 = 0$$

$$2(\bar{w}+w) + 3 = 0$$

$\Rightarrow 2(u - \cancel{iv} + u + \cancel{iv}) + 3 = 0$

$\rightarrow 4u + 3 = 0$

which is a straight line in the ω-plane.

5) Find the image of the triangular region in the z-plane bounded by the lines $x=0$, $y=0$ and $x+y=1$, under the transformation $\omega = 2z$.

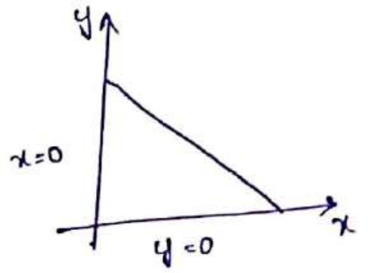
Solⁿ: Given transformation is $\omega = 2z$

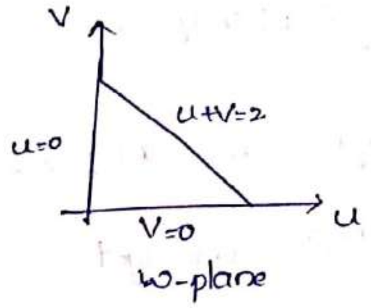
Let $z = x+iy$ & $\omega = u+iv$

then $u+iv = 2(x+iy)$

$u+iv = 2x + i2y$

i.e., $u = 2x$, $v = 2y$ $\Rightarrow x = \frac{u}{2}$ & $y = \frac{v}{2}$





When $x=0 \Rightarrow \frac{u}{2} = 0 \Rightarrow u = 0$

$y = 0 \Rightarrow \frac{v}{2} = 0 \Rightarrow v = 0$

and $x+y=1 \Rightarrow \frac{u}{2} + \frac{v}{2} = 1 \Rightarrow u+v=2$.

Thus the given map $\omega = 2z$ transforms the triangle in z-plane on to the triangle in the ω-plane bounded by $u=0$, $v=0$ and $u+v=2$.

6) Find and plot the rectangular region $0 \le x \le 1$, , $0 \le y \le 2$ under the transformation $w = \sqrt{2}\, e^{\frac{i\pi}{4}} z + (1-2i)$

Sol^n: Given transformation $w = \sqrt{2}\, e^{\frac{i\pi}{4}} z + (1-2i)$

Let $z = x+iy$, $w = u+iv$

$\therefore\ u+iv = \sqrt{2}\, e^{\frac{i\pi}{4}} (x+iy) + (1-2i)$

$= \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right](x+iy) + (1-2i)$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x+iy) + (1-2i)$

$= (1+i)(x+iy) + (1-2i)$

$= x + iy + ix - y + 1 - 2i$

$= (x-y+1) + i(x+y-2)$

$\therefore\ u+iv = (x-y+1) + i(x+y-2)$.

i.e., $u = x-y+1$ & $v = x+y-2$. —— ①

Putting $x=0$ in eq^n ①, we get

$x=0$, $u = -y+1$ & $v = y-2 \Rightarrow v = 1-u-2 \Rightarrow \boxed{v = -u-1}$ $[\because y = 1-u]$

Putting $x=1$ in eq^n ①, we get

$x=0$, $u = 1-y+1 = 2-y$ & $v = 1+y-2 = y-1$

$v = y-1 \Rightarrow v = 2-u-1$

$\boxed{v = 1-u}$

Putting $y=0$ in eq^n ①, we get

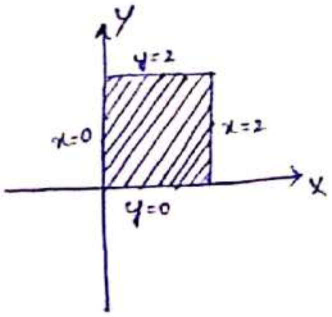
$y=0$, $u = x+1$ & $v = x-2 \Rightarrow v = u-1-2$, $\boxed{v = u-3}$

Putting $y=2$, $u = x-2+1 = x-1$

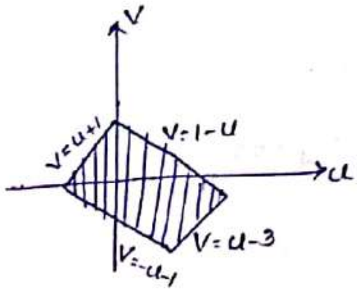
& $v = x+\cancel{2}-\cancel{2} = x$

$$\boxed{v = u+1}$$

Thus, the region is a rectangle bounded by the lines $v=-u-1$, $v=1-u$, $v=u-3$, $v=u+1$ that two regions are shown below.



z-plane



w-plane

7) Under the transformation $w = \frac{z-i}{1-iz}$ find the image of the circle $|z|=1$ in the w-plane

i) $|w|=1$

ii) $|z|$

$\frac{1-iz}{1+iz}$

9) Determine the image of the region interior to the hyperbola $x^2-y^2=2$ in z-plane under the transformation $w=z^2$

Sol^n: Given transformation $w=z^2$

Let $z=x+iy$ and $w=u+iv$

$$w=z^2$$

$$u+iv=(x+iy)^2$$

$$u+iv=(x^2-y^2)+i(2xy)$$

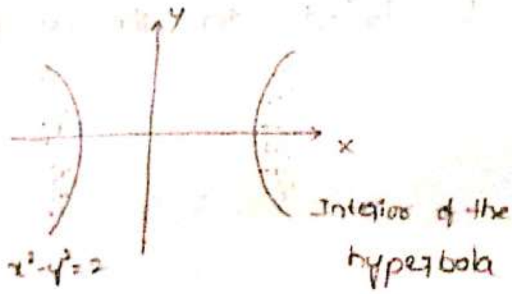
$$\therefore u=x^2-y^2, \quad v=2xy$$

Given hyperbola $x^2-y^2=2$.

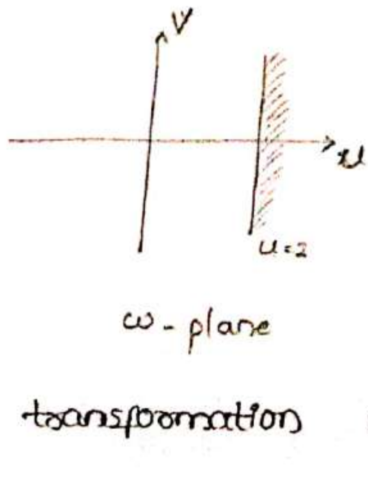
$\therefore u=2$

$u=2$ is a straight line which is parallel to the v axis.

The image of the interior of the hyperbola $x^2-y^2=2$ is the region to the right of the line $u=2$



z-plane



w-plane

10) Find the image of the line $x=4$ in the z-plane under the transformation $w=z^2$,

Sol^n: Given transformation $w = z^2$.

Let $z = x+iy$ and $w = u+iv$

$$w = z^2$$

$$\Rightarrow u+iv = (x+iy)^2$$

$$u+iv = (x^2-y^2)+i(2xy)$$

$$\therefore u = x^2-y^2 \quad \text{—①}, \quad v = 2xy \quad \text{—②}$$

To find the image of the line $x=4$, put $x=4$ in the eq^n ① & ②

$$① \Rightarrow u = 16-y^2 \quad \text{—③ and}$$

$$② \Rightarrow v = 8y$$

$$\Rightarrow y = \frac{v}{8} \quad \text{—④}$$

Sub $y = \frac{v}{8}$ in eq^n ③, we get

$$③ \Rightarrow u = 16 - \frac{v^2}{8^2}$$

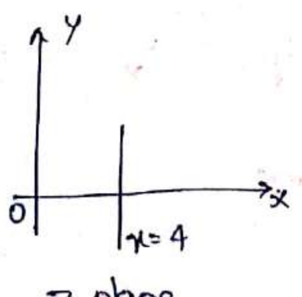
$$64u = (16)(64) - v^2$$

$$v^2 = (16)(64) - 64u = 16[64-4u]$$

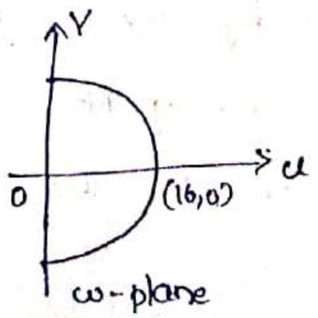
$$v^2 = 4^2[64\cancel{(1-4u)}]$$

$$\therefore v^2 = (-4)(4^2)(u-4^2)$$

This represents a parabola in w-plane with vertex $(16,0)$ and focus at the origin. and symmetrical about the real axis.



z-plane



w-plane

11) Find the image of the region in the z-plane $(y \geq 0)$ between the lines $y=0$ and $y=\pi/2$ under the transformation $w=e^z$.

Sol^n: Given transformation $w=e^z$.

Let $z=x+iy$ and $w=Re^{i\phi}$

so $Re^{i\phi}=e^{x+iy}$

$R.e^{i\phi}=e^x.e^{iy}$

i.e., $R=e^x$ —① and $\phi=y$ —②

[Given lines $y=0$ and $y=\pi/2$]

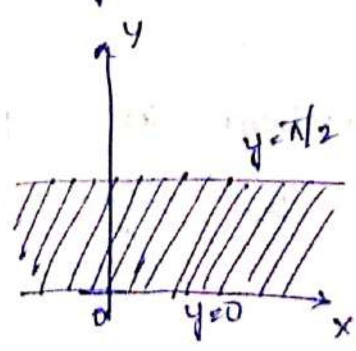
Now find the image b/w the lines $y=0$ & $y=\pi/2$

$y=0$, ② $\Rightarrow \phi=0$, represents radial line making an angle of 0 radians with the x-axis.

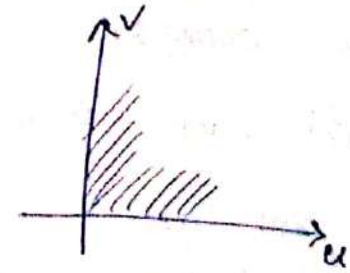
$y=\pi/2$, ② $\Rightarrow \phi=\pi/2$, represents radial line making an angle of $\pi/2$ radians with the x-axis.

Thus, when e^x increases from 0 to ∞ monotonically as x takes values from $-\infty$ to $+\infty$, the line $y=\pi/2$ in z-plane is mapped on to the ray $\phi=\pi/2$ excluded to the origin in w-plane.

Hence the infinite strip bounded by the lines $y=0$ and $y=\pi/2$ is mapped on to the



z-plane



w-plane

upper quadrant of ω-plane.

12) Find the image of the domain in z-plane to the left of the line $x=-3$ under the transformation $w=z^2$.

13) Show that the transformation $\omega=z^2$ maps the circle $|z-1|=1$ in to the cardioid $r=2(1+\cos\theta)$ where $\omega=re^{i\theta}$ in the ω-plane.

Bilinear Transformation (Mobius Transformation):-

The transformation $\omega=\frac{az+b}{cz+d}$ where a,b,c,d are complex constants and $ad-bc\neq 0$ is known as bilinear transfo-rmation. This is also known as Mobius (or) linear fractional transformation. The condition $ad-bc\neq 0$ is to ensure that the transformation is conformal, for $\frac{dw}{dz}=\frac{ad-bc}{(cz+d)^2}$ and if $ad-bc=0$, every point of the z-plane is a critical point.

Cross-ratio formula:-

Let z_1, z_2, z_3, z_4 be points in the z-plane which are mapped on to the points $\omega_1, \omega_2, \omega_3, \omega_4$ respectively under the bilinear transformation $w=\frac{az+b}{cz+d}$ then $\frac{(\omega_1-\omega_2)(\omega_3-\omega_4)}{(\omega_1-\omega_4)(\omega_3-\omega_2)}$

$$=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-x_2)}$$

Three given distinct points can also be mapped on to three prescribed points by one and only one, linear fractional tra-nsformation $\omega=f(z)$. This mapping is given simplicity by the eq^n. $\dfrac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)} = \dfrac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

i) Find the bilinear transformation which transforms the points $\infty, i, 0$ in the z-plane into $0, 1, \infty$ in the ω-plane.

Sol^n: Given $z_1 = \infty$, $z_2 = i$, $z_3 = 0$

$\omega_1 = 0$, $\omega_2 = 1$, $\omega_3 = \infty$

The required transformation is given by

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(\omega-0)(1-\infty)}{(\omega-\infty)(1-0)} = \frac{(z-\infty)(i-0)}{(z-0)(i-\infty)}$$

$$\Rightarrow \frac{\omega(1-\infty)}{1(\omega-\infty)} = \frac{(z-\infty)i}{z(i-\infty)}$$

$$\Rightarrow \frac{\omega(1-\infty)}{\cancel{1(\omega-\infty)}\ -1(\infty-\omega)} = \frac{i(z-\infty)}{-z(\infty-i)}$$

$$\Rightarrow \frac{-\omega(1-\infty)}{1(\infty-\omega)} = \frac{(-i)(z-\infty)}{z(\infty-i)}$$

$$\Rightarrow \left(\frac{-\omega}{1}\right) \underset{n\to\infty}{Lt} \left(\frac{1-n}{n-\omega}\right) = \left(\frac{-i}{z}\right) \underset{n\to\infty}{Lt} \left(\frac{z-n}{n-i}\right)$$

$$\Rightarrow \left(\frac{-\omega}{1}\right) \underset{n\to\infty}{Lt} \left(\frac{0-1}{1-0}\right) = \left(\frac{-i}{z}\right) \underset{n\to\infty}{Lt} \left(\frac{0-1}{1-0}\right)$$ [$\because \frac{\infty}{\infty}$ so using L-hospital rule]

$\Rightarrow \left(-\frac{\omega}{i}\right)(-1) = \left(\frac{-1}{z}\right)(-1)$

$$\frac{\omega}{i} = \frac{1}{z}$$

$$\omega = -\frac{1}{z}$$

$\therefore$ $\omega = -\frac{1}{z}$ is the required Bilinear transformation.